

The arrow of time, black holes, and quantum mixing of large N Yang-Mills theories

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ABSTRACT: Quantum gravity in an AdS spacetime is described by an $SU(N)$ Yang-Mills theory on a sphere, a bounded many-body system. We argue that in the high temperature phase the theory is intrinsically non-perturbative in the large N limit. At any nonzero value of the 't Hooft coupling λ , an exponentially large (in N^2) number of free theory states of wide energy range (or order N) mix under the interaction. As a result the planar perturbation theory breaks down. We argue that an arrow of time emerges and the dual string configuration should be interpreted as a stringy black hole.

KEYWORDS: Random Systems, AdS-CFT Correspondence, Statistical Methods, Black Holes in String Theory.

JHEP12(2007)027

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1. Introduction

While the equations of general relativity are time symmetric themselves, one often finds solutions with an intrinsic arrow of time, due to the presence of spacelike singularities. Familiar examples include FRW cosmologies and the formation of a black hole in a gravitational collapse. In the case of a gravitational collapse to form a black hole, the direction of time appears to be thermodynamic, since a black hole behaves like a thermodynamical system [1–3]. It has also long been speculated that the thermodynamic arrow of time observed in nature may be related to the big bang singularity [4].

In an anti-de Sitter (AdS) spacetime, a microscopic understanding of the emergence of thermodynamic behavior in a gravitational collapse can be achieved using the AdS/CFT correspondence [5–7], which states that quantum gravity in an asymptotic $AdS_5 \times S^5$ spacetime is described by an $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills (SYM) theory on an S^3 .

The classical gravity limit of the AdS string theory corresponds to the large N and large 't Hooft coupling limit of the Yang-Mills theory. A matter distribution of classical mass M in AdS, can be identified¹ with an excited state of energy $E = \mu N^2$ in the SYM theory with μ a constant independent of N . The gravitational collapse of the matter distribution can be identified with the thermalization of the corresponding state in SYM theory, with the resulting black hole² identified with thermal equilibrium [8, 9]. In this context, it is natural to suspect that the appearance of a spacelike singularity at the end point of a collapse should be related to certain aspect of thermalization in the SYM theory.³

A crucial element in the above description is the large N limit. $\mathcal{N} = 4$ SYM theory on S^3 is a closed, bounded quantum mechanical system with a discrete energy spectrum. At any finite N , no matter how large, such a theory is quasi-periodic (i.e. has recurrences), time reversible, and never really thermalizes. However, to match the picture of a gravitational collapse in classical gravity, an arrow of time should emerge in the large N limit for the SYM theory in a generic state of energy $E = \mu N^2$ with a sufficiently large μ . This consistency requirement immediately raises several questions:

1. What is the underlying physical mechanism for the emergence of an arrow of time in Yang-Mills theory?
2. Is large 't Hooft coupling needed?
3. Suppose an arrow of time also emerges at small 't Hooft coupling, what would be the bulk string theory interpretation of the SYM theory in such an excited state? A stringy black hole? Does such a stringy black hole have a singularity?
4. Is there a large N phase transition as one decreases the 't Hooft coupling from infinity to zero?

¹See appendix A for a brief review of parameter translations in AdS/CFT.

²Assume M is sufficiently big that a big black holes in AdS is formed, which also implies that μ should be sufficiently big.

³Some interesting ideas regarding spacelike singularities and thermalization have also been discussed recently in [10].

It would be very desirable to have a clear physical understanding of the above questions, which could shed light on how spacelike singularities appear in the classical limit of a quantum gravity and thus lead to an understanding of their resolution in a quantum theory.

The emergence of an arrow of time is also closely related to the information loss paradox.⁴ At finite N , the theory is unitary and there is no information loss. But in the large N limit, an arrow of time emerges and the information is lost, since one cannot recover the initial state from the final thermal equilibrium. Thus the information loss in a gravitational collapse is clearly a consequence of the classical approximation (large N limit), but not a property of the full quantum theory. While AdS/CFT in principle resolves the information loss paradox, it remains a puzzle whether one can recover the lost information using a semi-classical reasoning.⁵ From this perspective it would also be valuable to understand the various questions listed in the last paragraph.

The purpose of the paper is to suggest a simple mechanism for the emergence of an arrow of time in the large N limit and to initiate a statistical approach to understanding the quantum dynamics of a YM theory in highly excited states. In particular, we argue that the perturbative planar expansions break down for real-time correlation functions and that there is a large N “phase transition” at zero ’t Hooft coupling.⁶ We also argue that time irreversibility occurs for any nonzero value of the ’t Hooft coupling.

The plan of the paper is as follows. In section 2 we introduce the subject of our study: a family of matrix quantum mechanical systems including $\mathcal{N} = 4$ SYM on S^3 . We highlight some relevant features of the energy spectrum of these theories. Motivated by the classical mixing properties, we introduce observables which could signal time irreversibility. The simplest of them are real-time correlation functions at finite temperature, which describe non-equilibrium linear responses of the systems. The rest of the paper is devoted to studying these observables, first in perturbation theory, and then using a non-perturbative statistical method. In sec 3 we compute real-time correlation functions in perturbation theory. We find that at any finite order in perturbation theory, the arrow of time does not emerge. In sec 4 we argue that the planar perturbative expansion has a zero radius of convergence and cannot be used to understand the long time behavior of the system. In section 5 we give a simple physical explanation for the breakdown of perturbation theory. We argue that for any nonzero ’t Hooft coupling, an exponentially large (in N^2) number of free theory states of wide energy range (or order N) mix under the interaction. As a consequence small λ and long time limits do not commute at infinite N . In section 6 we develop a statistical approach to studying the dynamics of the theories in highly excited states, which indicates that time irreversibility occur for any nonzero ’t Hooft coupling λ . We conclude in section 7 with a discussion of implications of our results.

2. Prelude: theories and observables of interest

In this section we introduce the systems and observables we want to study.

⁴To our knowledge this connection was first pointed out in the context of AdS/CFT in [9].

⁵See e.g. [9, 11–17] for recent discussions.

⁶See [18, 19] for some earlier discussion of a possible large N phase transition in λ .

2.1 Matrix mechanical systems

We consider generic matrix quantum mechanical systems of the form

$$S = N \text{tr} \int dt \left[\sum_{\alpha} \left(\frac{1}{2} (D_t M_{\alpha})^2 - \frac{1}{2} \omega_{\alpha}^2 M_{\alpha}^2 \right) \right] - \int dt V(M_{\alpha}; \lambda) \quad (2.1)$$

which satisfy the following requirements:

1. M_{α} are $N \times N$ matrices and $D_t M_{\alpha} = \partial_t - i[A, M_{\alpha}]$ is a covariant derivative. One can also include fermionic matrices, but they will not play an important role in this paper and for simplicity of notations we suppress them.
2. The frequencies ω_{α} in (2.1) are nonzero for any α , i.e. the theory has a mass gap and a unique vacuum.
3. The number of matrices is greater than one and can be infinite. When there is an infinite number of matrices, we require the theory to be obtainable from a renormalizable field theory on a compact space.
4. $V(M_{\alpha}; \lambda)$ can be written as a sum of *single-trace* operators and is controlled by a coupling constant λ , which remains fixed in the large N limit.

$\mathcal{N} = 4$ SYM on S^3 is an example of such systems with an infinite number of matrices (including fermions) when the Yang-Mills and matter fields are expanded in terms of spherical harmonics on S^3 (see e.g. [20, 21]). In this case, ω_{α} are integer or half-integer multiples of a fundamental frequency $\omega_0 = 1/R$ with R the radius of the S^3 . The number of modes with frequencies $\omega_{\alpha} = \frac{k}{R}$ increases with k as a power. $V(M_{\alpha}; \lambda)$ can be schematically written as⁷

$$V = N \left(\sqrt{\lambda} V_3(M_{\alpha}) + \lambda V_4(M_{\alpha}) \right) \quad (2.2)$$

where V_3 and V_4 contain infinite sums of single-trace operators which are cubic and quartic in M_{α} and $\partial_t M_{\alpha}$. $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling.

In this paper we work in the large N limit throughout. Our discussion will only depend on the large N scaling of various physical quantities and not on the specific structure of the theories in (2.1) like the precise field contents and exact forms of interactions. For purpose of illustration, we will often use as a specific example the following simple system

$$S = \frac{N}{2} \text{tr} \int dt \left[(D_t M_1)^2 + (D_t M_2)^2 - \omega_0^2 (M_1^2 + M_2^2) - \lambda M_1 M_2 M_1 M_2 \right] . \quad (2.3)$$

⁷The precise form of the interactions depends on the choice of gauge. It is convenient to choose Coulomb gauge $\nabla \cdot \vec{A} = 0$, in which the longitudinal component of the gauge field is set to zero. In this gauge, M_{α} include also non-propagating modes coming from harmonic modes of ghosts and the zero component of the gauge field.

2.2 Energy spectrum

(2.1) has a $U(N)$ gauge symmetry and physical states are singlets of $U(N)$. One can classify energy eigenstates of a theory by how their energies scale with N in the large N limit. We will call the sector of states whose energies (as measured from the vacuum) are of order $O(1)$ the low energy sector. As motivated in the introduction, we are mainly interested in the sector of states whose energies are of order μN^2 with μ independent of N , which will be called the high energy sector. The density of states in the low energy sector is of order $O(1)$, i.e. independent of N , while that of the high energy sector can be written in a form

$$\Omega(E) \sim e^{s(\mu)N^2}, \quad E = \mu N^2 \tag{2.4}$$

with $s(\mu)$ some function independent of N . (2.4) follows from the fact that the number of ways to construct a state of energy of order $O(N^2)$ from $O(N^2)$ oscillators of frequency of $O(1)$ is an exponential in N^2 . The presence of interaction should not change this behavior at least for μ sufficiently large. (2.4) is the reason why we restrict to more than one matrix in (2.1). For a gauged matrix quantum mechanics with a single matrix one can reduce the matrix to its eigenvalues and (2.4) does not apply. When μ is sufficiently large, $s(\mu)$ should be a monotonically increasing function⁸ of μ and we will restrict our definition of high energy sector to such energies.

For $\mathcal{N} = 4$ SYM, states in the low energy sector correspond to fundamental string states in the AdS spacetime, while the states in the high energy sectors may be considered as black hole microstates.⁹

A convenient way to study a system in excited states is to put it in a canonical ensemble with a temperature $T = \frac{1}{\beta}$. The partition function and free energy are defined by (Tr denotes sum over all physical states and H is the Hamiltonian)

$$Z = \text{Tr} e^{-\beta H} = e^{-\beta F} . \tag{2.5}$$

We will always keep T fixed in the large N limit. Below low and high temperature refers to how the temperature is compared with the mass gap of a theory.¹⁰ As one varies T , different parts of the energy spectrum are probed. For the family of matrix quantum mechanical systems (2.1), there are two distinct temperature regimes. At low temperature, one probes the low energy sector and the free energy F is of order $O(1)$. At high temperature F is of order $O(N^2)$ and the high energy sector is probed. It may seem surprising at first sight that one can probe the sector of energies of $O(N^2)$ using a temperature of $O(1)$. This is due to the large entropy factor (2.4) which compensates the Boltzmann suppression. For $\mathcal{N} = 4$ SYM theory at strong coupling, there is a first order phase transition separating the two regimes at a temperature of order $1/R$, where R is the AdS radius [22, 6, 8]. A first order phase transition has also been found for various theories in the family of (2.1) at weak coupling [23, 24, 21].

⁸That is, the theory should have a positive specific heat for μ sufficiently large.
⁹Note that at a sufficiently high energy, the most entropic object in AdS is a big black hole.
¹⁰For example for $\mathcal{N} = 4$ SYM on S^3 , low (high) temperature means $T \ll \frac{1}{R}$ ($T \gg \frac{1}{R}$)

An important feature of the high energy sector is that the large N limit is like a thermodynamic limit with N^2 playing the analogous role of the volume factor. In this limit the number of degrees of freedom goes to infinity while the average excitations per degree of freedom remain finite. The thermal partition function

$$Z(\beta) = \text{Tr} e^{-\beta H} = \int dE \Omega(E) e^{-\beta E} \quad (2.6)$$

is sharply peaked at an energy $E_\beta \sim O(N^2)$ (with a width of order $O(N)$) determined by

$$\left. \frac{\partial S(E)}{\partial E} \right|_{E_\beta} = \beta, \quad S(E) = \log \Omega(E) \quad (2.7)$$

Note that the leading N dependence of $S(E)$ has the form $S(E) = N^2 s(\mu)$ (see (2.4)) with $\mu = E/N^2$ characterizing the average excitations per oscillator degree of freedom. Equation (2.7) can also be interpreted as the equivalence between canonical and micro-canonical ensemble.¹¹ Note that since $F \sim O(N^2)$, the high temperature phase can be considered a “deconfined” phase [25, 8].

2.3 Observables

In a classical Hamiltonian system, time irreversibility is closely related with the mixing property of the system, which can be stated as follows. Consider time correlation functions

$$C_{AB}(t) = \langle A(\Phi^t X) B(X) \rangle - \langle A \rangle \langle B \rangle \quad (2.8)$$

where A, B are functions on the classical phase space parameterized by X . $\Phi^t X$ describes the Hamiltonian flow, where Φ^t is a one-parameter group of volume-preserving transformations of the phase space onto itself. $\langle \dots \rangle$ in (2.8) denotes phase space average over a constant energy surface. The system is mixing¹² iff [26]

$$C_{AB}(t) \rightarrow 0, \quad t \rightarrow \infty \quad (2.9)$$

for any *smooth* L^2 functions A and B .

The closest analogue of (2.8) for the matrix quantum mechanical systems we are considering would be

$$G_i(t) = \langle i | \mathcal{O}(t) \mathcal{O}(0) | i \rangle - \langle i | \mathcal{O}(0) | i \rangle^2 \quad (2.10)$$

where $|i\rangle$ is a generic energy eigenstate in the high energy sector, and \mathcal{O} is an arbitrary gauge invariant operator which when acting on the vacuum creates a state of finite energy of order $O(1)$. More explicitly, denoting $|\psi_{\mathcal{O}}\rangle = \mathcal{O}(0) |\Omega\rangle$ with $|\Omega\rangle$ the vacuum, we require

¹¹In contrast in the low energy sector, since both the free energy and the density of states are independent of N , in generic models there is no large parameter that one can use to perform the saddle point approximation to equate two ensembles.

¹²Note that mixing is a stronger property than ergodic which involves long time average. The ergodic and mixing properties can also be characterized in terms of the spectrum of the Koopman operator. For example, a system is mixing iff the eigenvalue 1 is simply degenerate and is the only proper eigenvalue of the Koopman operator [26].

$\langle \psi_{\mathcal{O}} | H | \psi_{\mathcal{O}} \rangle \sim O(1)$. Note that for $\mathcal{N} = 4$ SYM on S^3 , a *local* operator $O(t, \vec{x})$ of dimension $O(1)$ on S^3 is not allowed by this criterion since $O(t, \vec{x})$ creates a state of infinite energy. To construct a state of finite energy one can smear the local operator over a spatial volume, e.g. by considering operators with definite angular momentum on S^3 . Without loss of generality, we can take \mathcal{O} to be of the form

$$\mathcal{O} = \text{tr}(M_{\alpha_1} \cdots M_{\alpha_{n_1}}) \text{tr}(M_{\beta_1} \cdots M_{\beta_{n_2}}) \cdots \text{tr}(M_{\gamma_1} \cdots M_{\gamma_{n_k}}) \quad (2.11)$$

with the total number of matrices $K = \sum_{i=1}^k n_k$ independent of N . We will call such operators small operators. The reason for restricting to small operators is that they have a well defined large N limit in the sense defined in [27]. More explicitly, if one treats the large N limit of a matrix quantum mechanics as a classical system, then (2.11) with $K \sim O(1)$ are *smooth* functions on the corresponding classical phase space. From AdS point of view, such operators correspond to fundamental string probes which do not deform the background geometry. If for all small operators \mathcal{O} and generic states $|i\rangle$ in the high energy sector

$$G_i(t) \rightarrow 0, \quad t \rightarrow \infty \quad (2.12)$$

one can say the system develops an arrow of time. In particular, (2.12) implies that one cannot distinguish different initial states from their long time behavior (i.e. information is lost).

Energy eigenstates are hard to work with. It is convenient to consider microcanonical or canonical averages of (2.10), for example, the thermal *connected* Wightman functions (see appendix B.1 for a precise definition of “connected” and the constant C below)

$$G_+(t) = \langle \mathcal{O}(t) \mathcal{O}(0) \rangle_{\beta} = \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \mathcal{O}(t) \mathcal{O}(0) \right) - C \quad (2.13)$$

or retarded functions

$$G_R(t) = \frac{1}{Z} \text{Tr} \left(e^{-\beta H} [\mathcal{O}(t), \mathcal{O}(0)] \right) . \quad (2.14)$$

We shall take the temperature T to be sufficiently high so that E_{β} determined from (2.7) lies in the high energy sector. Equation (2.12) implies that¹³

$$G_R(t) \rightarrow 0, \quad G_+(t) \rightarrow 0, \quad t \rightarrow +\infty . \quad (2.15)$$

Note that $G_R(t)$ measures the linear response of the system to external perturbations caused by \mathcal{O} . That $G_R(t) \rightarrow 0$ for $t \rightarrow \infty$ implies that any small perturbation of the system away from the thermal equilibrium eventually dies away. In a weaker sense than (2.12), (2.15) can also be considered as an indication of the emergence of an arrow of time.

In frequency space, the Fourier transform¹⁴ of (2.13) and (2.14) can be written in terms of a spectral density function $\rho(\omega)$ (see appendix B.1 for a review)

$$G_+(\omega) = \frac{1}{1 - e^{-\beta\omega}} \rho(\omega)$$

¹³(2.12) in fact implies the following to be true for any ensemble of states.

¹⁴We use the same letter to denote the Fourier transform of a function, distinguishing them by the argument of the function.

$$G_R(\omega) = - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega - \omega' + i\epsilon} \tag{2.16}$$

(2.15) may be characterized by properties of the spectral density $\rho(\omega)$. For example from the Riemann-Lebesgue theorem, (2.15) should hold if $\rho(\omega)$ is an integrable function on the real axis. Since other real-time correlation functions can be obtained from G_+ (or spectral density function $\rho(\omega)$) from standard relations, for the rest of the paper, we will focus on G_+ only.

For $\mathcal{N} = 4$ SYM at strong coupling, it is convenient to take \mathcal{O} to have a definite angular momentum l on S^3 . (2.13) and (2.14) can be studied by considering a bulk field propagating in an eternal AdS black hole geometry and one does find the behavior (2.15) as first emphasized in [9]. In the bulk language, (2.15) can be heuristically interpreted as the fact that any small perturbation of the black hole geometry eventually dies away by falling into the horizon. Furthermore, by going to frequency space, one finds that the Fourier transform $G_+(\omega, l)$ has a rich analytic structure in the complex ω -plane,¹⁵ which encodes that the bulk black hole geometry contains a horizon and singularities. The main features can be summarized as follows [28]:

1. $G_+(\omega, l)$ has a continuous spectrum with $\omega \in (-\infty, +\infty)$. This is due to the presence of the horizon in the bulk.
2. In the complex ω -plane, the only singularities of $G_+(\omega, l)$ are poles. The decay rate for $G_+(t)$ at large t is controlled by the imaginary part of the poles closest to the real axis, which is of order β .
3. The presence of black hole singularities in the bulk geometry is encoded in the behavior of $G_+(\omega, l)$ at the imaginary infinity of the ω -plane.¹⁶ In particular,
 - 3a. $G_+(\omega, l)$ decays exponentially as $\omega \rightarrow \pm i\infty$.
 - 3b. Derivatives of $G_+(\omega, l)$ over l evaluated at $l = 0$ are divergent as $\omega \rightarrow \pm i\infty$.

As emphasized in [28], none of the above features survives at finite N , in which case¹⁷

$$G_+(\omega) = 2\pi \sum_{m,n} e^{-\beta E_m} \rho_{mn} \delta(\omega - E_n + E_m)$$

has a discrete spectrum and is a sum of delta functions supported on the real axis. This indicates that concepts like horizon and singularities only have an approximate meaning in a semi-classical limit (large N limit).

¹⁵Similar things can also be said about $G_R(\omega, l)$ which can be obtained from $G_+(\omega, l)$ using standard relations.

¹⁶See also [29] for signature of the black hole singularities in coordinate space.

¹⁷Note that even though $\mathcal{N} = 4$ SYM on S^3 is a field theory, at finite N the theory can be effectively considered as a theory with a finite number of degrees of freedom, since for any given energy E , there are only a finite number of modes below that energy. Furthermore, given that the number of modes with frequency $\frac{k}{R}$ grows with k only as a power, it is more entropically favorable to excite modes with low k for $E \sim O(N^2)$ and modes with $\omega_\alpha \sim O(N)$ are almost never excited.

To understand the information loss paradox and the resolution of black hole singularities, we need to understand how and why they arise in the classical limit of a quantum gravity. In Yang-Mills theory, this boils down to understanding what physics is missed in the large N limit and why missing it is responsible for the appearance of singularities and the loss of information. With these motivations in mind, in this paper we are interested in understanding the following questions

1. Can one find a qualitative argument for the emergence of an arrow of time in the large N limit?
2. Does the analytic behavior observed at strong coupling persist to weak coupling?

which we turn to in the following sections.

3. Non-thermalization in perturbation theory

In this section we consider (2.13) in perturbation theory in the planar limit. We will find that real-time correlation functions have a discrete spectrum and oscillatory behavior. Thus the theory does not thermalize in the large N limit.

In perturbation theory, $G_+(t)$ can be computed using two methods. In the first method, one computes $G_E(\tau)$ with $0 < \tau < \beta$ in Euclidean space using standard Feynman diagram techniques. $G_+(t)$ can then be obtained by taking $\tau = it + \epsilon$. An alternative way is to double the fields and use the analogue of the Schwinger-Keldysh contour to compute the Feynman function $G_F(\omega)$ in frequency space [30], from which $G_+(\omega)$ can be obtained. In the Euclidean-time method it is more convenient to do the computation in coordinate space since one does not have to sum over discrete frequencies, while in the real-time method frequency space is more convenient to use.

We look at the free theory first.

3.1 Free theory

To evaluate (2.13) in free theory, it is convenient to use the Euclidean method. The Euclidean correlator

$$G_E^{(0)}(\tau) = \langle \mathcal{O}(\tau) \mathcal{O}(0) \rangle_{0,\beta}, \quad 0 \leq \tau < \beta \tag{3.1}$$

with \mathcal{O} of the form (2.11) can be computed using the Wick contraction¹⁸

$$\underbrace{M_{ij}^{\alpha_1}(\tau) M_{kl}^{\alpha_2}(0)} = \frac{\delta_{\alpha_1 \alpha_2}}{N} \sum_{m=-\infty}^{\infty} g_E^{(0)}(\tau - m\beta; \omega_{\alpha_1}) U_{il}^{-m} U_{kj}^m \tag{3.2}$$

where $g_E^{(0)}$ is the propagator at zero temperature

$$g_E^{(0)}(\tau; \omega) = \frac{1}{2\omega} e^{-\omega|\tau|}. \tag{3.3}$$

¹⁸see e.g. [31] for a derivation of the following equation and some examples of correlation functions in free theory.

In (3.2) U is a unitary matrix which arises due to covariant derivatives in (2.1) and can be understood as the Wilson line of A wound around the τ direction. In the evaluation of free theory correlation functions $\langle \cdots \rangle_{0,\beta}$ in (3.1), one first performs the Wick contractions (3.2) and then performs the unitary matrix integral over U , which plays the role of projecting the intermediate states to the singlet sector. In the large N limit, the U integral can be evaluated by a saddle point approximation. Note in particular that [24]

$$U \rightarrow 1, \quad T \rightarrow \infty \tag{3.4}$$

Equation (3.4) indicates that the singlet condition should not play an important role for states of sufficiently high energy.

For definiteness, we now restrict to theories with a single fundamental frequency ω_0 like $\mathcal{N} = 4$ SYM or (2.3). Relaxing this restriction does not affect our main conclusions, as will be commented on in various places below. Wick contractions in (3.1) give rise to terms of the form $e^{n\omega_0\tau}$ for some integer n , while the U -integral computes the coefficients of these terms. Thus (3.1) always has the form

$$G_E^{(0)}(\tau) = \sum_{n=-\Delta}^{\Delta} c_n(\beta) e^{n\omega_0\tau} \tag{3.5}$$

where Δ is the dimension of the operator.¹⁹ Analytically continuing (3.5) to real time, we find that

$$G_+^{(0)}(t) = \sum_{n=-\Delta}^{\Delta} c_n(\beta) e^{-in\omega_0 t} \tag{3.6}$$

and

$$G_+^{(0)}(\omega) = 2\pi \sum_{n=-\Delta}^{\Delta} c_n(\beta) \delta(\omega - n\omega_0) . \tag{3.7}$$

Thus in the large N limit, the correlation function always shows a discrete spectrum is quasi-periodic. The results are generic. If the theory under consideration has several incommensurate fundamental frequencies, one simply includes a sum like those (3.5) and (3.7) for each such frequency. The maximal number of independent exponentials is 2^K , where K is the total number of matrices in \mathcal{O} . This is due to that each matrix in \mathcal{O} can only connect states with a definite energy difference.

It is also instructive to obtain (3.6) using a different method. By inserting a complete set of free theory energy eigenstates in (2.13) we find that

$$G_+^{(0)}(t) = \frac{1}{Z_0} \sum_{a,b} e^{-\beta\epsilon_a} \rho_{ab} e^{i(\epsilon_a - \epsilon_b)t} \tag{3.8}$$

where $|a\rangle$ is a free theory state with energy ϵ_a and $\rho_{ab} = |\langle a|\mathcal{O}(0)|b\rangle|^2$. To understand the structure of (3.8) we expand $\mathcal{O}(0)$ in terms of creation and annihilation operators associated with each $(M_\alpha)_{ij}$, from which we find that

¹⁹Note that for $\mathcal{N} = 4$ SYM the dimension of M_α is given by $\frac{\omega_\alpha}{\omega_0}$. For other matrix quantum mechanical systems without conformal symmetry one can use a similar definition in free theory. For bosonic operators, Δ are integers.

- A. Due to energy conservation, \mathcal{O} can connect levels whose energy differences lie between $-\Delta\omega_0$ and $\Delta\omega_0$, i.e. ρ_{ab} can only be non-vanishing for $|\epsilon_a - \epsilon_b| < \Delta\omega_0$.
- B. \mathcal{O} can only connect states whose energy differences are integer multiples of ω_0 i.e. ρ_{ab} can only be non-vanishing for $\epsilon_a - \epsilon_b = n\omega_0$ with $|n| < \Delta$ integers (or half integers if \mathcal{O} is fermionic). Similarly, in the cases where \mathcal{O} contains K types of matrices of different frequency ω_i it can only connect states whose energy differs by $\sum_{i=1}^K n_i\omega_i$, where n_i are integers whose absolute values are bounded by the number of matrices of each type appearing in \mathcal{O} .

As a result, (3.8) must have the form (3.6). Note that the argument based on (3.8) applies not only to the thermal ensemble, but in fact to correlation functions in any density matrix (or pure state).

To summarize, one finds that in free theory a real-time thermal two-point function always has a discrete spectrum and is quasi-periodic in the large N limit. This implies that once one perturbs the theory away from thermal equilibrium, the system never falls back and keeps oscillating. This is not surprising since the system is free and there is no interaction to thermalize any disturbance. Note that this is distinctly different from the behavior (2.15) found at strong coupling. In particular, this implies that the bulk description of the high temperature phase in free theory looks nothing like a black hole. Also note that the story here is very different from that of the orbifold CFT in the AdS₃/CFT₂ correspondence. There the mass gap in free theory goes to zero in the large N limit in the long string sector [32]. As a result, one finds that free theory correlation functions in the long string sector do resemble those from a BTZ black hole [9, 33].

3.2 Perturbation theory

In this subsection we use a simple example (2.3) for illustration. The general features discussed below apply to generic theories in (2.1) including $\mathcal{N} = 4$ SYM.

In perturbation theory $G_E(\tau)$ can be expanded in terms of λ as

$$G_E(\tau) = \sum_{n=0}^{\infty} \lambda^n G_E^{(n)}(\tau) \tag{3.9}$$

where $G_E^{(0)}$ is the free theory result. We will be only interested in the connected part of $G_E(\tau)$. Higher order corrections are obtained by expanding $e^{-\lambda \int d\tau V}$ in the path integral with V given by the quartic term in (2.3). More explicitly, a typical contribution to $G_E^{(n)}(\tau)$ in (3.9) has the form

$$\frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \langle \mathcal{O}(\tau) \mathcal{O}(0) V(\tau_1) \cdots V(\tau_n) \rangle_{\beta,0} \tag{3.10}$$

The free theory correlation function inside the integrals in (3.10) can be computed by first using Wick contraction (3.2) and then doing the U integral. The general structure of (3.10) can be summarized as follows:

1. The planar diagram contribution to $G_E^{(n)}(\tau)$ scales like N^0 , while diagrams of other topologies give higher order $1/N^2$ corrections. The number R_n of planar diagrams grows like a power in n , i.e. is bounded by C^n with C some finite constant [34].
2. The τ -integrations are over a compact segment and are all well defined. A typical term in (3.10) after the integration has the structure

$$g_{kj}^{(n)}(\beta)\tau^l e^{k\omega_0\tau} \tag{3.11}$$

where l and k are integers. l can take values from 0 to n , while k from $-2n - \Delta$ to $2n + \Delta$ where Δ is the dimension of \mathcal{O} in free theory.

Analytically continuing (3.9) to Lorentzian time by taking $\tau = it + \epsilon$, we find

$$G_+(t, \lambda) = \sum_{n=0}^{\infty} \lambda^n G_+^{(n)}(t) \tag{3.12}$$

where typical terms in $G_+^{(n)}(t)$ have the t -dependence of the form

$$g_{kl}^{(n)}(\beta)t^l e^{ik\omega_0 t} \tag{3.13}$$

with the range of l and k given after equation (3.11). After Fourier transforming to frequency space we find that at each order in the perturbative expansion $G_+^{(n)}(\omega)$ (and thus the spectral density function $\rho(\omega)$) consists of sums of terms of the form

$$g_{kl}^{(n)}\delta^{(l)}(\omega - k\omega_0) \tag{3.14}$$

where the superscript l denotes the number of derivatives.

If the theory has more than one fundamental frequencies, since the interaction vertices are traces of a finite number of matrices they only connect states with definite energy differences. More and more frequencies will appear in the spectrum of a correlation function as we go to higher and higher orders in the perturbative expansion. The increase in the number of frequencies is exponential in the order of the expansion but at any fixed finite order no matter how large the spectrum of the correlation functions is discrete.

One origin of t^l terms in (3.13) is the shifting of frequency from the free theory value. For example, suppose the free theory frequency is shifted to $\omega = \omega_0 + \lambda\omega_1 + \dots$, one would get terms of the form (3.13) when expanding the exponential $e^{i\omega t}$ in λ . One can in principle improve the perturbation theory by resumming such contributions using Dyson's equations. However, there appears no systematic way of doing this for a composite operator (3.4). In appendix B.2, we prove that real-time correlation functions of fundamental modes M_α again have a discrete spectrum in the improved perturbative expansion.

4. Break down of planar perturbation theory

It is well known that at zero temperature the planar expansion of a matrix quantum mechanics has a finite radius of convergence in the λ -plane (see e.g. [34] for a recent

discussion and earlier references). If this persists at finite temperature, properties of the theory at zero coupling or in perturbation theory should hold at least for the coupling constant being sufficiently small. In particular, from our discussion of last section, one would conclude that real-time correlation functions for generic gauge invariant operators should be quasi-periodic and an arrow of time does not emerge at small 't Hooft coupling. In this section, we argue that the planar perturbative expansion in fact breaks down for real-time correlation functions and thus perturbation theory cannot be used to understand the long-time behavior of the system at any nonzero coupling.

From our discussion in section 3.2, we expect the Euclidean correlation function (3.9) should have a finite radius of convergence for any given $\tau \in (0, \beta)$. After analytic continuation to real time, the convergence of the expansion in Euclidean time implies that (3.12) should have a finite radius $\lambda_c(t)$ of convergence for any given t . However, it does not tell how $\lambda_c(t)$ changes with t in the limit $t \rightarrow \infty$. In this section we argue that the radius of convergence goes to zero in the large t limit. Note that the convergence of the perturbative expansion depends crucially on how $g_{kl}^{(n)}$ in (3.14) fall off with n . We will argue below that the falloff is slow enough that perturbation theory breaks down in the long time limit. In frequency space, one finds that n -th order term in the expansion grows like $n!$.²⁰

We will again use (2.3) as an illustration. The argument generalizes immediately to generic systems in (2.1). For simplicity, we will consider the high temperature limit (3.4) in which we can replace U in (3.2) by the identity matrix, e.g.

$$\underbrace{M_{1ij}(\tau) M_{1kl}(0)} = \frac{1}{N} \sum_{m=-\infty}^{\infty} g_E^{(0)}(\tau - m\beta; \omega_0) \delta_{il} \delta_{kj} = \frac{1}{N} \delta_{il} \delta_{kj} g_E(\tau; \omega) \quad (4.1)$$

where

$$g_E(\tau; \omega) = \frac{1}{2\omega} (e^{-\omega\tau}(1 + f(\omega)) + e^{\omega\tau}f(\omega)), \quad \tau \in (0, \beta) \quad (4.2)$$

with

$$f(\omega) = \frac{1}{e^{\beta\omega} - 1}. \quad (4.3)$$

Note that outside the range in (4.2), $g_E(\tau)$ is periodic.

For our purpose it is enough to examine the Wightman function for M_1 ,

$$D_+(t) = \frac{1}{Z(\beta)} \text{Tr} \left(e^{-\beta H} M_1(t) M_1(0) \right). \quad (4.4)$$

An exactly parallel argument to that of the last section leads to the expansion

$$D_+(t, \lambda) = \sum_{n=0}^{\infty} \lambda^n D_+^{(n)}(t) \quad (4.5)$$

where typical terms in $D_+^{(n)}(t)$ have the t -dependence of the form

$$d_{kl}^{(n)}(\beta) t^l e^{ik\omega_0 t} \quad (4.6)$$

²⁰Note that in frequency space the relation between real-time and Euclidean correlation functions is not simple, since Euclidean correlation functions are only defined at discrete imaginary frequencies.

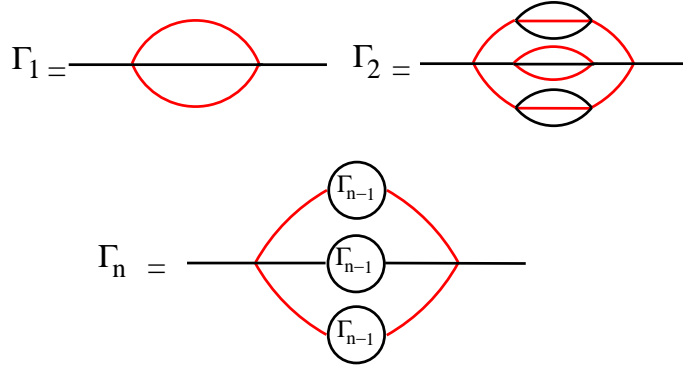


Figure 1: A family of diagrams which indicates that the perturbation theory break down in the long time limit. Black and red lines denote propagators of M_1 and M_2 respectively.

The convergence of series depends on how $d_{kl}^{(n)}$ fall off with n . For our purpose it is enough to concentrate on the term with the highest power t in each order, i.e. the coefficients of t^n with given k . More explicitly, we will look at a term of the form

$$D_+(t, \lambda) = D_+^{(0)}(t) \sum_{n=0}^{\infty} c_n \lambda^n t^n + \dots \quad (4.7)$$

where $D_+^{(0)}$ is the free theory expression.

As before we will first compute (4.7) in Euclidean time and then perform an analytic continuation. Calculating

c_n explicitly at each loop order for all n is of course impractical. Our strategy is as follows. We will identify a family (in fact infinite families as we will see below) of planar Feynman diagrams of increasing loop order and show that their contribution to c_n falls off like a power in n . Barring any unforeseen magical cancellation,²¹ this would imply that the perturbation series (3.12) has a zero radius of convergence in the $t \rightarrow \infty$ limit. The simplest set of diagrams which meet our purpose are given by figure 1.

These graphs appear at orders $d_1 = 2, d_2 = 8, d_3 = 26, \dots$ of perturbation theory where

$$d_i = 3d_{i-1} + 2 = 3^i - 1, \quad i = 1, 2, \dots \quad (4.8)$$

We denote the contribution of each diagram by $\Gamma_i(\tau)$. For our purpose, it is not necessary to compute the full graph. We will only need to calculate the term in each graph with the highest power of τ , i.e. the term proportional to τ^{d_i} . Also note that in each diagram, the symmetry factor is exactly 1. Let us start with Γ_1 , which is given by

$$\Gamma_1(\tau - \tau') = \lambda^2 \int_0^\beta d\tau_1 d\tau_2 g_E(\tau - \tau_1; \omega_0) g_E^3(\tau_1 - \tau_2; \omega_0) g_E(\tau_2 - \tau'; \omega_0) \quad (4.9)$$

Note the identity

$$g_E^3(\tau; \omega_0) = \frac{3}{(2\omega_0)^2} f^2(\omega_0) \left(e^{\beta\omega_0} g_E(\tau; \omega_0) + \frac{f(\omega_0)}{f(3\omega_0)} g_E(\tau; 3\omega_0) \right) \quad (4.10)$$

²¹Note that since we are in the high temperature phase, in which supersymmetry is badly broken, there is no obvious reason for suspecting such magical cancellations.

Now plug (4.10) into (4.9). It is easy to convince oneself that the term proportional to $g_E(\tau; 3\omega_0)$ in (4.10) will not generate a term proportional to τ^2 and we will ignore it. The contribution of the term proportional to $g_E(\tau; \omega_0)$ can be found by noting the identity²²

$$\int_0^\beta d\tau_1 d\tau_2 g_E(\tau - \tau_1; \omega_0) g_E(\tau_1 - \tau_2; \omega_0) g_E(\tau_2 - \tau'; \omega_0) = \frac{1}{2} \frac{\partial^2}{(\partial\omega_0^2)^2} g_E(\tau - \tau'; \omega_0) \quad (4.11)$$

The right hand side of (4.11) contains a piece $\frac{1}{2} \frac{(\tau - \tau')^2}{(2\omega_0)^2} g_E(\tau - \tau')$ plus parts with smaller powers of $\tau - \tau'$. Thus the term in (4.9) proportional to $(\tau - \tau')^2$ is given by

$$\Gamma_1(\tau - \tau') = \frac{\alpha\lambda^2}{2} (\tau - \tau')^2 g_E(\tau - \tau') + \dots \quad (4.12)$$

where

$$\alpha = \frac{3f(1+f)}{(2\omega_0)^4}, \quad f = f(\omega_0) \quad (4.13)$$

The term proportional to τ^{d_i} for higher order diagrams $\Gamma_i(\tau)$ can now be obtained by iterating the above procedure. A useful identity is

$$\begin{aligned} \int_0^\beta d\tau_1 d\tau_2 g_E(\tau - \tau_1; \omega_0) g_E(\tau_1 - \tau_2; \omega_0) (\tau_1 - \tau_2)^n g_E(\tau_2 - \tau'; \omega_0) \\ = \frac{(\tau - \tau')^{n+2}}{(2\omega_0)^2} \frac{1}{(n+2)(n+1)} g_E(\tau - \tau'; \omega_0) + \dots \end{aligned} \quad (4.14)$$

where we kept only the term with the highest power of $\tau - \tau'$, as lower power terms will not contribute to the terms in which we are interested. We find that the term proportional to τ^{d_i} in $\Gamma_i(\tau)$ is given by

$$\Gamma_i(\tau) = F_i \lambda^{d_i} \tau^{d_i} g_E(\tau; \omega_0) + \dots \quad (4.15)$$

where F_i satisfy the recursive relation

$$F_{i+1} = F_i^3 \frac{\alpha}{d_{i+1}(d_{i+1} - 1)}. \quad (4.16)$$

Thus F_i can be written as

$$F_i = \alpha^{\frac{d_i}{2}} \Lambda_i \quad (4.17)$$

with

$$\Lambda_i = \prod_{k=0}^{i-1} \left(\frac{1}{d_{i-k}(d_{i-k} - 1)} \right)^{3^k} \quad (4.18)$$

Λ_i in the large i limit can be easily estimated and we find

$$\Lambda_i \approx e^{-\frac{3}{2}d_i}, \quad i \gg 1.$$

²²If two matrices have different frequencies in the product $g_E(\tau; \omega_0) g_E^2(\tau; \omega_1)$ there is also a term proportional to $g_E(\tau; \omega_0)$ with coefficient $\frac{1}{(2\omega_1)^2} f(\omega_1)(1+f(\omega_1))$ and the rest of the analysis follows with minor changes.

Summing all our diagrams together and analytically continuing to Lorentzian time with $\tau = it + \epsilon$, we find that²³

$$\sum_i \Gamma_i(t) \approx D_+^{(0)}(t) \sum_{i=1}^{\infty} (-1)^i \left(\frac{\lambda t}{h_c}\right)^{d_i} + \dots \quad (4.19)$$

with h_c given by

$$h_c = \frac{e^{\frac{3}{2}}}{\sqrt{\alpha}} = \frac{e^{\frac{3}{2}}(2\omega_0)^2}{\sqrt{3f(1+f)}} \quad (4.20)$$

Equation (4.19) implies that the radius of convergence in λ is given by

$$\lambda_c(t) \sim \frac{1}{t} \quad (4.21)$$

which goes to zero as $t \rightarrow \infty$.

It is also instructive to repeat the computation of figure 1 in frequency space using the real-time method. The calculation is straightforward and we will only summarize the result. One finds that the contribution of Γ_i to the Feynman function $D_F(\omega)$ grows like $d_i!$. Thus one expects that the perturbative expansion in frequency space is not well defined for any frequency. Note that the non-analyticity in frequency space can be expected since in going to frequency space one has to integrate the full real time-axis and the Fourier transform is sensitive to the long time behavior. Also note that the $n!$ factorial behavior in perturbation theory often implies an essential singularity at $\lambda = 0$ (see also below).

We conclude this section with some remarks:

1. In the zero-temperature limit $h_c \rightarrow \infty$ and the set of terms in (4.19) all go to zero.
2. To simplify our discussion, we have only considered diagrams in figure 1. There are in fact many other diagrams of similar type contributing at other orders in λ . For example, by including those in figure 2, one can get contributions for all even orders in λ rather than only (4.8). The qualitative conclusion we reached above is not affected by including them.²⁴
3. By taking in consideration the diagrams on the left of figure 2 the sum in (4.19) is extended ²⁵ to all even powers of λt and is oscillating therefore the singularities in λt should lie on the imaginary axis. Let us suppose that for a given λ , $D_+(t)$ has a singularity in t at q_c/λ with q_c lying in the upper half plane.²⁶ Now Fourier transforming $D_+(t)$ we find that

$$D_+(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} D_+(t) \quad (4.22)$$

²³Since we are only interested in the asymptotic behavior of the sum for large i we have replaced F_i by its asymptotic value.

²⁴There are also potentially an infinite number of other sets of diagrams which can lead to the behavior (4.21), e.g. one can replace Γ_1 by any diagram whose highest power in t is the same as the order of perturbation and then iterates.

²⁵the value of h_c also changes

²⁶Note that q_c^*/λ must also be a singularity of $D_+(t)$.

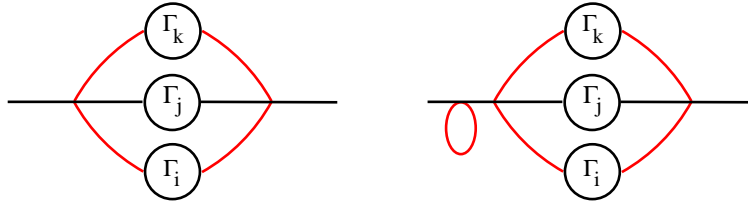


Figure 2: By including the diagrams on the left with all possible $i, j, k \geq 0$ we can get contribution at every even order of λ instead of (4.8). Γ_0 denotes a single propagator. Diagrams on the right can also contribute to the odd orders if (2.3) contains additional interactions of the form $\text{tr}A^2B^2$.

The presence of q_c/λ and q_c^*/λ implies $D_+(\omega)$ contains a term of the form for $\omega > 0$

$$D_+(\omega) \sim e^{i\omega \frac{q_c}{\lambda}} \quad (4.23)$$

Thus $D_+(\omega)$ contains an essential singularity at $\lambda = 0$.

4. The $n!$ behavior in perturbative expansion in frequency space (say in the computation of $D_F(\omega)$) arises from a *single* class of Feynman diagrams. This is reminiscent of renormalons in field theories [35–37]. In particular, when Borel resumming the divergent series, depending on whether ω is greater or smaller than ω_0 , the singularities on the Borel plane can appear on the positive or negative real axis,²⁷ also reminiscent of the IR and UV renormalons.
5. Note that in the limit $T \rightarrow \infty$, h_c in (4.20) scales with T as $h_c \sim \frac{\omega_0^3}{T}$, i.e.

$$\lambda_c(t) \sim \frac{\omega_0^3}{tT} \quad (4.24)$$

For fixed λ , we expect a singularity for $D_+(t)$ at

$$t \sim \frac{\omega_0^3}{\lambda T} \quad (4.25)$$

Note that the right hand side of (4.25) is reminiscent of the magnetic mass scale for a Yang-Mills theory (see e.g. [38]). However, in our matrix quantum mechanics, there is no infrared divergence and it is not clear whether there is a connection.

6. The discussion can be straightforwardly applied to a generic theory in (2.1) with cubic and quartic couplings. In fact the argument also applies to a single anharmonic oscillator at finite temperature, even though in that case one does not expect the perturbative expansion to converge anyway.²⁸ Similarly, the argument also applies to a single-matrix quantum mechanics if one does not impose the singlet condition. When imposing the singlet condition, the matrix U in equation (3.2) cannot be set to

²⁷Since we only have contributions to even order in λ , we cannot make a conclusion from our discussion so far.

²⁸See appendix D for further elaborations on the example of a single anharmonic oscillator and a discussion on the differences between the single anharmonic oscillator and the matrix systems under consideration.

1 and our argument does not apply. Similarly our argument does not apply to (2.1) in the low energy sector, in which U is always important. Indeed using the results of [31, 39], one can show that to leading order in the large N limit, correlation functions at finite temperature can be written in terms of those at zero temperature and we do expect that the perturbation theory has a finite radius of convergence.

7. Our argument indicates that perturbation theory breaks down in the long time limit for a generic theory in (2.1). However, for any specific theory (say $\mathcal{N} = 4$ SYM theory) we cannot rule out magical cancelations which could in principle make the coefficients of n -th order term much smaller than indicated by the diagrams we find. If magical cancelations do occur in some theory, that would also be extremely interesting since it indicates some hitherto unknown hidden structure.²⁹

5. Physical explanation for the breakdown of planar expansion

In this section we give an alternative argument for the breakdown of perturbation theory in the long time limit, which complements that of last section. The discussion below should apply to a generic theory in (2.1). For definiteness we use $\mathcal{N} = 4$ SYM as an illustration example.

We first set up some notations. We write the full Hamiltonian as

$$H = H_0 + V(\lambda) \tag{5.1}$$

with H_0 the Hamiltonian of the free theory and V the interaction. We denote a free theory energy eigenstate by $|a\rangle$ with energy ϵ_a . $|0\rangle$ is the (unique) free theory vacuum. The energy eigenstates of the interacting theory H are denoted by $|i\rangle$ with energy E_i . $|\Omega\rangle$ is the interacting theory vacuum. We can expand

$$|i\rangle = \sum_a c_{ia} |a\rangle \tag{5.2}$$

with c_{ia} satisfying

$$\sum_a |c_{ia}|^2 = \sum_i |c_{ia}|^2 = 1 . \tag{5.3}$$

We first recall some relevant features of the free theory energy spectrum of $\mathcal{N} = 4$ SYM on S^3 . Since ω_α in (2.1) are all integer or half-integer multiples of $\omega_0 = \frac{1}{R}$, the free theory energy spectrum is quantized in units $\frac{1}{2}\omega_0$. Typical energy levels are degenerate. The degeneracy is of $O(1)$ in the low energy sector and of order $e^{O(N^2)}$ in the high energy sector. The exponentially large degeneracy in the high energy sector can be seen as follows. From (2.7) the density of states $\Omega_0(\epsilon)$ in the high energy sector is of order $e^{O(N^2)}$. Since the energy levels are equally spaced with spacings order $O(1)$, it must be that typical energy levels have a degeneracy of order $e^{O(N^2)}$. Alternatively, the number of ways to construct

²⁹Since we are working at a finite temperature, supersymmetry alone should not be sufficient for the cancelations.

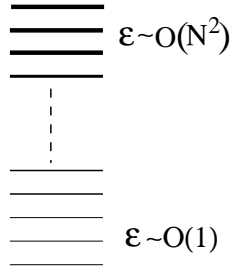


Figure 3: The energy spectrum of free $\mathcal{N} = 4$ SYM on S^3 is quantized. Typical degeneracy for an energy level $\epsilon \sim O(1)$ is of order $O(1)$. Typical degeneracy for a level of energy $\epsilon \sim O(N^2)$ is of order $e^{O(N^2)}$.

a state of energy of order $O(N^2)$ from $O(N^2)$ oscillators of frequency of $O(1)$ is clearly exponentially large in N^2 .

When a theory contains $n > 1$ incommensurate fundamental frequencies, the free theory spectrum at energies of order $O(N^2)$ will have level spacings of order $O(N^{-2(n-1)})$. Since the density of states are of order $e^{O(N^2)}$, the degeneracy of a typical state is again of order $e^{O(N^2)}$ as in the case of one fundamental frequency. That the level spacings go to zero as a negative power in N does not change our conclusion of previous sections regarding the thermalization in free theory or at any finite order in perturbation theory, since as emphasized there a small operator can only connect states whose energy differences are of order $O(N^0)$. Therefore such small level spacings cannot be accessed dynamically. The restriction above to a finite number of incommensurate fundamental frequencies is not essential. The conclusion applies to any theories in which the number of fundamental frequencies increases only as a power of the frequency. For these theories, the thermal ensemble is dominated by states built from oscillators whose frequencies are smaller than or of order of β^{-1} . Thus the effective number of fundamental frequencies is finite.

Now let us turn on the interaction $V(\lambda)$ (2.2) with a tiny but nonzero λ . We will focus on the high energy sector. Given that free theory energy levels are highly degenerate, one would like to apply degenerate perturbation, say to diagonalize V in a degenerate subspace of energy $E \approx \mu N^2$ and of dimension $e^{O(N^2)}$. For this purpose we need to choose a basis in the degenerate subspace. This is a rather complicated question, due to difficulties in imposing singlet conditions.³⁰ However, when μ is sufficiently large we expect the singlet condition not to play an important role.³¹ So to simplify our discussion we will ignore the singlet condition below. A convenient orthonormal basis of energy eigenstates for H_0 are then monomials of various oscillators (appropriately normalized), i.e.

$$\prod_{\alpha} \prod_{i,j=1}^N \left(M_{\alpha ij}^{\dagger} \right)^{n_{\alpha ij}} |0\rangle . \quad (5.4)$$

In the basis (5.4), if the full theory is not integrable, V can be effectively treated as

³⁰The trace relations are important for states of such energies.

³¹As remarked earlier, in the high temperature limit the saddle point for U (in (3.2)) approaches the identity matrix.

an (extremely) sparse random matrix.³² The sparseness is due to that each term in V can connect a given monomial state to at most N^k other states, where k is an $O(1)$ number.³³ Randomness has to do with the large dimension of the subspace and to the fact that there is no preferred ordering for the states within the same subspace. Diagonalizing V , we thus expect, from general features of a sparse random matrix (see appendix C for a summary),

1. The degeneracy of the free theory will generically be broken.³⁴
2. A number of states of order $e^{O(N^2)}$ will mix under the perturbation.
3. The typical level spacing between energy levels should be proportional to the inverse of the density of states and is thus exponentially small, of order $e^{-O(N^2)}$. Note that this is exponentially smaller than the level spacings in a free theory with a finite number of incommensurate fundamental frequencies.

The story is in fact a little more intricate. We expect the degenerate perturbation to be a good guide if the spread of energy eigenvalues after diagonalizing V in a subspace is smaller than the spacings between nearby energy levels. The spread Γ of eigenvalues of V can be estimated by (see appendix C)

$$\Gamma^2 \sim \Gamma_a^2 = \sum_{a \neq b} |\langle a|V|b \rangle|^2 \sim O(N^2) \tag{5.5}$$

for any nonzero λ , where the sum restricts to a degenerate subspace. Note that (5.5) only depends on that V is a single trace operator and does not depend on the specific structure of it. That $\Gamma \sim O(N)$ implies that it is not sufficient to diagonalize V within a degenerate subspace. It appears more appropriate to diagonalize³⁵ it in a subspace with energy spread of order $O(N)$. Thus in addition, we expect that:

4. an interacting theory eigenstate $|i\rangle$ is strongly coupled to free theory states $|a\rangle$ within an energy shell of order $O(N)$.

This statement will be justified in the next section from a somewhat different perspective. That $\Gamma \sim O(N)$ for any nonzero λ in the 't Hooft limit indicates a tiny λ may not really be considered as a small perturbation after all.

Various features discussed above when turning on a small λ are clearly non-perturbative in nature. However, it may be hard to probe them directly using Euclidean space observables like partition functions and Euclidean correlation functions. These observables probe only average behaviors within an energy difference range of order $O(T)$ or larger and thus

³²Here we restrict V to a single energy level. When including all energy levels V is banded and sparse. The banded structure is due to energy conservation.

³³This is because each term in V is a monomial of a few matrices.

³⁴For Yang-Mills theories on S^3 , there are remaining degeneracies associated with the isometry group $SO(4)$ of S^3 . Except when one considers the sectors with very large angular momenta on S^3 , typical representations of $SO(4)$ are rather small and should not affect our general argument.

³⁵This statement is of course only heuristic since there is no sharp criterion to decide what should be the precise size of the subspace. However, we expect the N scaling should be robust.

may not be sensitive to the changes in level spacings at smaller scales.³⁶ In contrast, real-time correlation functions are much more sensitive. For example, consider the Lehmann spectral decomposition of $G_+(t)$, i.e.

$$\begin{aligned} G_+(t) &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \mathcal{O}(t) \mathcal{O}(0) \right) \\ &= \frac{1}{Z} \sum_{i,j} e^{-\beta E_i + i(E_i - E_j)t} |\langle i | \mathcal{O}(0) | j \rangle|^2 \end{aligned} \tag{5.6}$$

where we have inserted complete sets of energy eigenstates $|i\rangle$ of the interacting theory. From (5.6), it is clear that $G_+(t)$ can in principle probe any small energy differences, provided one takes t to be large enough. This explains the breakdown of perturbation theory in the long time limit observed in $G_+(t)$. At large N , the $\lambda \rightarrow 0$ and $t \rightarrow \infty$ limits do not commute.

In this and the last sections we have presented two lines of largely independent arguments that suggest that planar perturbation theory breaks down in the long time limit. The first argument (last section) is based on an honest Feynman diagram calculation, which establishes the breakdown of perturbation theory, but does not tell us directly whether the spectral functions are continuous or not. The second argument (this section), based on the energy spectrum, is more heuristic, but implies that the spectral functions are continuous in the large N limit. There are reasons to believe that the two arguments should be closely related. As stressed earlier, the breakdown of perturbation theory from the class of Feynman diagrams considered in the last section only happens at a sufficiently high temperatures, at which the thermal ensemble is dominated by states of energy of order $O(N^2)$ and the energy spectrum becomes quasi-continuous.³⁷ Nevertheless, a precise relation between the two arguments is not clear at this point, as they use very different languages (one Feynman diagrams and the other energy levels). It would be very desirable to find a direct connection and to have an understanding of the time scale (4.25) from the point of view of the energy levels.

6. A statistical approach

The argument of section 4 shows that the planar perturbation theory breaks down in the large time limit, but it does not tell us what the long time behavior is. Non-perturbative tools are needed to understand the long time behavior of real time correlation functions in the large N limit. Here we develop a statistical approach, taking advantage of the extremely large density of states in the high energy sector. In this section we outline the main idea and the results, leaving detailed calculations to various appendices. The statistical approach

³⁶Of course if one is able to compute Euclidean observables exactly, one should be able to extract all the interesting physics. After all, real-time observables can be obtained from Euclidean ones by analytic continuation. It is just often the case that real-time physics is encoded in a very subtle way in Euclidean observables.

³⁷Applied to the theory at zero temperature or at a temperature below the deconfinement temperature, the class of diagrams gives a convergent contribution. At such a temperature, the thermal ensemble is dominated by states of energy of order $O(N^0)$, which are not quasi-continuous when turning on interactions.

enables us to derive some qualitative features satisfied by the Wightman function for a generic operator at finite temperature, including that it has a continuous spectral density function and should decay to zero in the long time limit. The features we find here are also shared by the Wightman function at strong coupling found from supergravity analysis.

Our starting point is the Lehmann spectral decomposition of $G_+(t)$ (5.6),

$$G_+(t) = \frac{1}{Z} \sum_{i,j} e^{-\beta E_i + i(E_i - E_j)t} \rho_{ij} \tag{6.1}$$

where

$$\rho_{ij} = |\langle i | \mathcal{O}(0) | j \rangle|^2 = |\mathcal{O}_{ij}|^2 \tag{6.2}$$

In momentum space

$$G_+(\omega) = \frac{1}{Z} \sum_{i,j} e^{-\beta E_i} \delta(\omega + E_i - E_j) \rho_{ij} . \tag{6.3}$$

Matrix elements \mathcal{O}_{ij} can in turn be expressed in terms of those of free theory using (c_{ia} was introduced in (5.2))

$$\mathcal{O}_{ij} = \langle i | \mathcal{O}(0) | j \rangle = \sum_{a,b} c_{ia}^* c_{jb} \langle a | \mathcal{O} | b \rangle = \sum_{a,b} c_{ia}^* c_{jb} \mathcal{O}_{ab} \tag{6.4}$$

where we have inserted complete sets of free theory states and $\mathcal{O}_{ab} = \langle a | \mathcal{O}(0) | b \rangle$.

Since for sufficiently high temperature, the sums in (6.3) and (6.4) are peaked at an energy with an extremely large density of states, one should be able to obtain the qualitative behavior of ρ_{ij} and $G_+(\omega)$ from statistical properties of \mathcal{O}_{ab} and c_{ia} . As discussed in the last section, in the interacting theory, we expect typical level spacings scale with N like $e^{-O(N^2)}$. In the large N limit, E_i can be considered as taking continuous values. Note that this by itself does not imply that $G_+(\omega)$ has a continuous spectral decomposition, since it is possible that ρ_{ij} only has support for states with finite energy differences. We argue below that ρ_{ij} has nonzero support between states with any $E_i - E_j \in (-\infty, \infty)$, which is independent of N , and thus $G_+(\omega)$ does have a continuous spectrum.

Let us first look at the statistical behavior of c_{ia} . For this purpose, consider the following density functions

$$\rho_a(E) = \sum_i |c_{ia}|^2 \delta(E - E_i) \tag{6.5}$$

$$\chi_i(\epsilon) = \sum_a |c_{ia}|^2 \delta(\epsilon - \epsilon_a) \tag{6.6}$$

$\rho_a(E)$, first introduced by Wigner [40], is also called the local spectral density function or strength function in the literature.³⁸ Using normalization properties of c_{ia} , one finds that

$$\int dE \rho_a(E) = 1, \quad \int d\epsilon \chi_i(\epsilon) = 1 \tag{6.7}$$

³⁸These density functions have been frequently used in quantum chaos literature, see e.g. [41]

$\rho_a(E)$ can be considered as the distribution of interacting theory eigenstates of energy E coupling to a free theory state $|a\rangle$. Similarly, $\chi_i(\epsilon)$ gives the distribution of free theory states of energy ϵ coupling to an exact eigenstate $|i\rangle$. The mean and the variances of the two distributions are given by

$$\bar{E}_a = \int dE E \rho_a(E) = \langle a|H|a\rangle \tag{6.8}$$

$$\sigma_a = \Gamma_a^2 = \int dE (E - \bar{E}_a)^2 \rho_a(E) = \sum_{b \neq a} |\langle a|V|b\rangle|^2 \tag{6.9}$$

$$\bar{\epsilon}_i = \int d\epsilon \epsilon \chi_i(\epsilon) = E_i - \langle i|V|i\rangle \tag{6.10}$$

$$\Sigma_i = \Delta_i^2 = \int d\epsilon (\epsilon - \epsilon_i)^2 \chi(E_i, \epsilon) = \sum_{j \neq i} |\langle i|V|j\rangle|^2 \tag{6.11}$$

\bar{E}_a and Γ_a give the center and the spread of interacting theory energy eigenstates coupling to a free state $|a\rangle$. Similarly, $\bar{\epsilon}_i$ and Δ_i give the center and the spread of free theory states coupling to an interacting theory energy eigenstate $|i\rangle$. Γ_a can be considered as a measure of correlation among energy levels of the interacting theory (since states whose energies differ by Γ_a could couple to the same free theory state and are thus correlated). Δ_i characterizes the range of free theory states which are mixed by perturbation. Note that the heuristic discussion after equation (5.5) implies that $\Delta_i \sim O(N)$, which we will confirm below using a different method.

Individual energy eigenstates are rather hard to work with. We will consider micro-canonical averages of (6.5) and (6.6). After all, for (6.1) and (6.3) we only need the behavior of ρ_{ij} averaged over states of similar energies. We will denote the average³⁹ of $\chi_i(\epsilon)$ over interacting theory states of energy E by $\chi_E(\epsilon)$ and similarly the average of $\rho_a(E)$ over free theory states $|a\rangle$ of similar energy ϵ by $\rho_\epsilon(E)$. Since the averages involve a huge number of states and the large N limit is like a thermodynamic limit in the high energy sector, we will assume that $\chi_E(\epsilon)$ is a smooth *slow* function⁴⁰ of E , i.e. it depends on E only through E/N^2 . Similarly $\rho_\epsilon(E)$ is assumed to depend on ϵ only through ϵ/N^2 . The center and variance of $\chi_E(\epsilon)$ and $\rho_\epsilon(E)$ will be denoted by $\bar{\epsilon}(E)$, $\Sigma(E) = \Delta^2(E)$, $\bar{E}(\epsilon)$, and $\sigma(\epsilon) = \Gamma^2(\epsilon)$ respectively.⁴¹ These quantities should also be slow functions of E or ϵ as they inherit the property from $\chi_E(\epsilon)$ and $\rho_\epsilon(E)$. In the appendix E we estimate these quantities and find that

$$\bar{\epsilon}(E) = N^2 g(\lambda, E/N^2)$$

³⁹More explicitly, the average can be written as

$$\chi_E(\epsilon) = \frac{1}{\Omega(E)} \sum_{E_i \in (E-\delta, E+\delta)} \chi_i(\epsilon) \tag{6.12}$$

where δ is small enough that $\Omega(E)$ does not vary significantly in the range $(E - \delta, E + \delta)$.

⁴⁰Note that a function $f(E)$ is considered a slow function if it can be written in a form $f(E) = N^a g(E/N^2)$, where $g(x)$ is a function independent of N .

⁴¹which can also be obtained by the average of various quantities (6.8)–(6.11) to leading order in large N .

$$\begin{aligned}
 \Sigma(E) &= N^2 h(\lambda, E/N^2) \\
 \bar{E}(\epsilon) &= N^2 \tilde{g}(\lambda, \epsilon/N^2) \\
 \sigma(\epsilon) &= N^2 \tilde{h}(\lambda, \epsilon/N^2)
 \end{aligned}
 \tag{6.13}$$

We emphasize that the large N scalings above only depend that V is given by N times single trace operators. Given that the underlying theory is not integrable and the extremely large number of states, we will thus approximate c_{ia} for fixed i as a random unit vector which centers at $\bar{\epsilon}_i$ with a spread of order $\Delta_i \sim O(N)$.

Now we turn to the statistical properties of \mathcal{O}_{ab} . Our earlier discussion for V in the free state basis (5.4) can be carried over to any operator \mathcal{O} of dimension $O(1)$. Thus \mathcal{O}_{ab} can be considered as an sparse banded random matrix. The matrix is banded since from energy conservation \mathcal{O} can only connect states whose energy difference is smaller than the dimension of \mathcal{O} . Note that even though \mathcal{O}_{ab} is sparse, for each row (or column), the number of nonzero entries grows with N as a power.

To summarize, we will assume the following statistical properties for c_{ia} and \mathcal{O}_{ab} :

1. For a given i , c_{ia} is a random unit vector with support inside an energy shell of width $O(N)$. In particular, the c_{ia} satisfy the same distribution for $|a\rangle$ of the same energy.
2. \mathcal{O}_{ab} is banded sparse random matrix, with the number of nonzero entries growing with N as a power.

Now consider any two states $|i\rangle$ and $|j\rangle$, with energies E_i and E_j respectively, for which $\omega = E_i - E_j \sim O(1)$. One finds that $\bar{\epsilon}_i - \bar{\epsilon}_j \sim O(1)$ and the energy shells of the two states overlap significantly. Given that the number of nonzero entries in a row or column of \mathcal{O}_{ab} grows with N as a power and that each element of c_{ia} satisfies the same distribution, one concludes from (6.4) that \mathcal{O}_{ij} should have support for any $\omega = E_i - E_j \sim O(1)$ and $G_+(\omega)$ has a continuous spectrum for $\omega \in (-\infty, +\infty)$. Note that the fact that $\Delta \sim O(N)$ is crucial for having a continuous spectrum $\omega \in (-\infty, +\infty)$. Suppose $\Delta \sim O(1)$, the spectrum cannot extend to $\pm\infty$ due to energy conservation.

One can further work out more detailed properties of ρ_{ij} . Leaving the detailed calculation in various appendices, we find that (after averaging ρ_{ij} over states of similar energies)

$$\rho_{E_1 E_2} = \frac{1}{\Omega(E)} A(\omega; E) = e^{-S(E)} A(\omega; E)
 \tag{6.14}$$

where $\Omega(E)$ and $S(E) = \log \Omega(E)$ are the density of states and entropy of the interacting theory respectively and

$$E = \frac{E_1 + E_2}{2}, \quad \omega = E_1 - E_2 .$$

Equation (6.14) is derived in appendix G along with properties of $A(\omega; E)$ stated below. Some useful formulas used in the derivation are collected in appendix F. $A(\omega; E)$ can be expressed in terms of an integral of $\chi_E(\epsilon)$ and $\bar{\epsilon}(E)$ (see equations (G.3) and (G.8)) and satisfies the following properties:

1. $A(\omega; E)$ is an even function of ω , i.e.

$$A(-\omega; E) = A(\omega; E) \tag{6.15}$$

2. As $\omega \rightarrow \infty$

$$A(\omega; E) \propto e^{-\frac{1}{2}\beta(E)|\omega|}, \quad \beta(E) = \frac{\partial S(E)}{\partial E} \tag{6.16}$$

3. $A(\omega, E)$ is integrable along the real axis and can at most have integrable singularities of the form

$$A(\omega; E) \propto \frac{1}{|\omega - \omega_s|^{\alpha_s}}, \quad \alpha_s < 1. \tag{6.17}$$

4. $A_E(\omega)$ depends on E only through E/N^2 , i.e. it can be written as

$$A(\omega; E) = A(\omega; \mu), \quad \mu = \frac{E}{N^2} \tag{6.18}$$

and A is a function independent of N .

Note that property 2 implies that in the large N limit, $\rho_{E_1 E_2} \sim 0$ for $E_1 - E_2 \sim N^a$ with $a > 0$.

The expression for $G_+(\omega)$ in momentum space can now be obtained by plugging (6.14) into (6.3) and using a saddle point approximation. We find that

$$\begin{aligned} G_+(\omega) &= \frac{1}{Z} \int dE e^{-\beta E} e^{S(E)+S(E+\omega)} e^{-S(E+\omega/2)} A(\omega, E/N^2) \\ &= \frac{1}{Z} \int dE e^{-\beta E+S(E)} \left[e^{S(E+\omega)-S(E+\omega/2)} A(\omega, E/N^2) \right] \\ &= e^{\frac{\beta\omega}{2}} A(\omega, \mu_\beta) \end{aligned} \tag{6.19}$$

where

$$\mu_\beta = \frac{E_\beta}{N^2}, \quad \left. \frac{\partial S(E)}{\partial E} \right|_{E_\beta} = \beta. \tag{6.20}$$

Note that since in the large N limit, E can be treated as continuous and ρ_{ij} has support for any energy difference, it is appropriate to approximate the sum in (6.3) by an integral. Also from the second line to the third line we have used that the quantity inside the bracket depends on E slowly and performed a saddle point approximation. We conclude this section with some remarks:

1. $G_+(\omega)$ has a continuous spectrum with $\omega \in (-\infty, +\infty)$ in the large N limit (note that ω does not scale with N).
2. Since $A(\omega, \mu)$ can at most have integrable singularities of the form (6.17) on the real axis, after a Fourier transform to coordinate space, $G_+(t)$ must go to zero in the limit $t \rightarrow \infty$. If $A(\omega; \mu)$ is a smooth function on the real axis, then $G_+(t)$ must decay exponentially with time.

3. Considering the last line of (6.19) as a definition for $A(\omega; \mu)$, for $\mathcal{N} = 4$ SYM on S^3 at strong coupling, the corresponding $A(\omega; \mu)$ can be found by solving the Laplace equation for a scalar field in an AdS black hole geometry and be expressed in terms of boundary values of renormalizable wave functions for the scalar field [28]. In particular, $A(\omega; \mu)$ found at strong coupling satisfy all the properties (6.15)–(6.18) (it is a smooth function on the real axis).
4. It should be possible to obtain an explicit expression for $A(\omega; \mu)$ (and thus $G_+(\omega)$) using the expressions found in the appendices (e.g. equation (G.3)) if one can find the density functions (6.5) and (6.6) for a sparse banded random matrix with varying density of states. While those for constant density of states have been discussed in the literature (see e.g. [41]), not much appears to be known for the non-constant density of states [42].

7. Discussions

In this paper we first showed that in perturbation theory, real-time correlation functions in the high temperature phase of (2.1) have a discrete spectrum and the system does not thermalize when perturbed away from thermal equilibrium. We then argued that the perturbative expansions for real-time correlation functions break down in the long time limit. The breakdown of perturbation theory indicates that at large N the $\lambda \rightarrow 0$ and $t \rightarrow \infty$ limits do not commute. The reason for the breakdown is that a wide energy range (of order $O(N)$) of degenerate free theory energy eigenstates mix under the interaction. The level spacings in the energy spectrum of $O(1)$ in the free theory become $e^{-O(N^2)}$. As a result, real-time correlation functions develop a continuous spectrum for any nonzero λ . The continuous spectrum was argued from a statistical approach developed in section 6, where we also show that real-time correlation functions should decay to zero as $t \rightarrow \infty$ and the system becomes time irreversible.

We should emphasize that our arguments in this paper are qualitative in nature and far from foolproof. For example, instead of being a random vector, c_{ia} could have some structure (e.g. being very sparse) within the range of its spread, in which case our statistical argument will not be valid.

It is also important to emphasize our results only apply to the high energy sector and in the low energy sector (or in the low temperature), there is no indication of breakdown of the planar expansion. In particular the results we describe here are not inconsistent with that the sector near the vacuum might be integrable in the large N limit [43]. In fact the results of [31, 39] are consistent with that the interpolation between the free theory and strong coupling may be smooth in the low temperature phase.

Our results indicate that there is a large N “phase transition” at $\lambda = 0$, i.e. physical observables undergo qualitative changes in the limit $\lambda \rightarrow 0$. The “phase transition” we find here is somewhat unusual, since it is not manifest in the Euclidean quantities like the partition function. The partition function appears to be smooth in the $\lambda \rightarrow 0$ limit. The “phase transition” is in real-time correlation functions and their Fourier transforms.

Real-time correlation functions decay to zero at large time at any finite λ , while oscillatory for $\lambda = 0$. In frequency space there is an essential singularity at $\lambda = 0$.

It would be interesting to understand whether one can continue the physics at small λ to large λ . If there is no further large N “phase transition” in λ , we expect that the analytic structure of various correlation functions observed at strong coupling should also be present at small λ . Such structure include the signatures of black hole singularities [29, 28] and the bulk-cone singularities [44].

Given that an arrow of time emerges for small λ in the large N limit, it is natural to ask what should be the string theory interpretation of the high temperature phase for $\mathcal{N} = 4$ SYM on S^3 at weak coupling, or from the microcanonical point of view, what is the bulk interpretation for a generic state in the high energy sector.

From the parameter relations in AdS/CFT,

$$\frac{l_s^2}{R^2} = \frac{1}{\sqrt{\lambda}}, \quad \frac{G_N}{R^8} = \frac{1}{N^2}, \quad G_N = l_p^8 \sim g_s^2 l_s^8, \quad . \quad (7.1)$$

one might conclude that at weak coupling $\lambda \ll 1$, $l_s \gg R$, i.e. the string length l_s is much bigger than the AdS curvature radius R . However, it seems unlikely one can give an invariant meaning to the statement. For example, even starting with a metric with $R \ll l_s$, one could perform a field redefinition of the form $g_{\mu\nu} \rightarrow g_{\mu\nu} + \alpha' R_{\mu\nu} + \dots$. In terms of new metric one then has $R \sim l_s$. Thus it seems to us that even for $\lambda \ll 1$, the corresponding bulk string theory should describe a spacetime of stringy scale, rather than sub-stringy scale. This is also expected from the gauge theory point of view. At weak coupling the only mass scale is the inverse radius of the sphere and there are no other lighter degrees of freedom. Thus the string scale has to be of the same order as that of the AdS curvature scale.

Can one interpret the bulk configuration corresponding to the high temperature phase at weak coupling as a stringy black hole? It seems to us the answer is likely to be yes. Let us list the properties that the corresponding bulk configuration should satisfy as expected from gauge theory, assuming there is no further large N “phase transition” between weak and strong couplings:

1. The bulk configuration should have an entropy and free energy of order $O(1/g_s^2)$.
2. The object absorbs all fundamental probes (since boundary correlation functions decay with time).
3. The bulk geometry should have a horizon (since the boundary theory has a continuous spectrum).
4. The bulk configuration is likely to have singularities (since the signatures of the black hole singularities in gauge theory at strong coupling cannot disappear as the coupling is changed if there is no phase transition).
5. A generic matter distribution will collapse into such a configuration (since in the boundary theory, a generic initial state will approach the thermal equilibrium).

6. Results in [24, 23] indicate that the Euclidean time circle in the dual geometry for the theory in the high temperature phase should become contractible.⁴²

From the properties above, it seems appropriate to call it a stringy black hole.

Finally let us mention that it is possible that a stronger version of equation (6.14) holds, i.e. for two generic states $|i\rangle, |j\rangle$ in the high energy sector,

$$\rho_{ij} = \frac{1}{\Omega(E)} A(\omega; E) \mathcal{R}_{ij}, \quad (7.2)$$

with

$$E = \frac{E_i + E_j}{2}, \quad \omega = E_i - E_j$$

and \mathcal{R}_{ij} a random matrix. Equation (7.2) is considered to be the hallmark of quantum chaos [45, 46].⁴³ Thus it is possible that $\mathcal{N} = 4$ SYM is chaotic in the high energy sector.⁴⁴ Such a chaotic behavior, if it exists, might be related to the BKL behavior near a spacelike singularity [50].

Acknowledgments

We would like to thank O. Aharony, T. Banks, W. Fischler, S. Hartnoll, V. Hubeny, A. Lawrence, J. McGreevy, M. Rangamani, A. Scardicchio, S. Shenker, M. Srednicki, L. Susskind for useful conversations. This work is supported in part by Alfred P. Sloan Foundation and U.S. Department of Energy (D.O.E) OJI grant, and funds provided by the U.S. Department of Energy (D.O.E) under cooperative research agreement #DF-FC02-94ER40818.

A. Parameter relations in AdS/CFT

$\mathcal{N} = 4$ super Yang-Mills theory is a conformally invariant theory with two parameters: the rank of gauge group N and the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$. From the operator-state correspondence, physical states of the theory on S^3 can be obtained by acting with gauge invariant operators on the vacuum and their energies are given by the conformal dimensions of the corresponding operators.

It was conjectured in [5] that $\mathcal{N} = 4$ SYM gives a nonperturbative description of type IIB superstring theory in $AdS_5 \times S_5$. The AdS string theory also has two parameters: the ratio between string length l_s and the curvature radius R of AdS, and the ratio between the (10d) planck length l_p and R . These ratios respectively characterize classical stringy

⁴²Note that this alone cannot imply that the bulk geometry is a black hole since even at zero coupling the time circle becomes contractible. As we argued earlier in this paper real-time correlation functions in free theory do not behave like those of a black hole.

⁴³It has also been argued in [46] that if (7.2) holds, then thermalization always occurs.

⁴⁴Chaos in a classical Yang-Mills theory was discussed before in [47]. Possible pole of quantum chaos in AdS/CFT and in black hole physics has also been discussed before in [48, 49].

corrections and quantum gravitational corrections beyond the classical supergravity. For small l_s/R and l_p/R , parameters of SYM theory and bulk string theory are related by⁴⁵ [9]

$$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}, \quad \frac{G_N}{R^8} = \frac{1}{N^2}, \quad G_N = l_p^8 \sim g_s^2 \alpha'^4. \quad (\text{A.1})$$

The above relations indicate that the classical supergravity limit is given by the large N and large λ limit of the SYM theory. In particular, a departure from the large N limit of the Yang-Mills theory corresponds to turning on *quantum* gravitational corrections in the AdS spacetime, while a departure from the large λ limit (with $N = \infty$) corresponds to turning on *classical* stringy corrections.

AdS/CFT implies an isomorphism between the Hilbert space of the two theories. In particular, any bulk configuration with asymptotic AdS_5 boundary conditions can be associated with a state (pure or density matrix) of the Yang-Mills theory. The mass M of the bulk configuration is related to the energy E in YM theory as [7, 6]

$$E \sim MR. \quad (\text{A.2})$$

Depending on how E scales with N in the large N limit, states of Yang-Mills theory are related to different objects in string theory in AdS. For example those whose E do not scale with N (i.e. of order $O(1)$) should correspond to fundamental string states. An object in AdS with a classical mass M satisfies

$$G_N M = \text{fixed}, \quad G_N/R^8 \rightarrow 0 \quad (\text{A.3})$$

From (A.1) and (A.2) the corresponding state in YM theory should have $E \sim O(N^2)$.

B. Self-energy in the real time formalism

In this appendix we first review some basic properties of real-time correlation functions. We then prove that the spectral density functions of fundamental fields in (2.1) have a discrete spectrum after the resummation of the self-energy diagrams à la Dyson.

B.1 Analytic properties of various real-time functions

Various real-time thermal Wightman function for an operator \mathcal{O} are defined by

$$\begin{aligned} G_+(t) &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \mathcal{O}(t) \mathcal{O}(0) \right) - C \\ G_-(t) &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \mathcal{O}(0) \mathcal{O}(t) \right) - C \\ G_F(t) &= \theta(t) G_+(t) + \theta(-t) G_-(t), \\ G_R(t) &= i\theta(t) \frac{1}{Z} \text{Tr} \left(e^{-\beta H} [\mathcal{O}(t), \mathcal{O}(0)] \right), \\ G_A(t) &= -i\theta(-t) \frac{1}{Z} \text{Tr} \left(e^{-\beta H} [\mathcal{O}(t), \mathcal{O}(0)] \right) \end{aligned} \quad (\text{B.1})$$

⁴⁵We omit order one numerical constants.

where Z is the partition function and C is a constant to be specified below. It is also convenient to introduce

$$G_{12}(t) = G_+(t - i\beta/2) \tag{B.2}$$

which can be obtained from $G_+(t)$ by an analytic continuation.

By inserting complete sets of states in (B.1), $G_+(t)$ can be written as

$$G_+(t) = \frac{1}{Z} \sum_{i \neq j} e^{-iE_j t} e^{iE_i(t+i\beta)} \rho_{ij} \tag{B.3}$$

where i, j sum over the physical states of the theory and $\rho_{ij} = |\langle i | \mathcal{O}(0) | j \rangle|^2$. Comparing (B.3) and (B.1), C is chosen to be

$$C = \frac{1}{Z} \sum_i e^{-E_i \beta} \rho_{ii} \tag{B.4}$$

Note that C is chosen so that the Fourier transform of $G_+(t)$ does not have a “contact” term proportional to $\delta(\omega)$. Assuming the convergence of the sums is controlled by the exponentials, it follows from (B.3) that $G_+(t)$ is analytic in t within the range $-\beta < \text{Im } t < 0$. Similarly $G_-(t)$ is analytic for $0 < \text{Im } t < \beta$ and $G_{12}(t)$ for $-\frac{\beta}{2} < \text{Im } t < \frac{\beta}{2}$.

Introducing the spectral density function

$$\rho(\omega) = (1 - e^{-\beta\omega}) \sum_{i,j} (2\pi) \delta(\omega - E_i + E_j) e^{-\beta E_j} \rho_{ij} \tag{B.5}$$

then the Fourier transforms of (B.1) can be written as

$$\begin{aligned} G_+(\omega) &= \frac{1}{1 - e^{-\beta\omega}} \rho(\omega) \\ G_{12}(\omega) &= e^{-\frac{1}{2}\beta\omega} G_+(\omega) = e^{\frac{1}{2}\beta\omega} G_-(\omega) = \frac{1}{2\sinh \frac{\beta\omega}{2}} \rho(\omega) \\ G_R(\omega) &= - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega - \omega' + i\epsilon} \\ G_A(\omega) &= - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega - \omega' - i\epsilon} \\ G_F(\omega) &= G_R(\omega) + iG_-(\omega) \end{aligned} \tag{B.6}$$

From (B.6) we also have

$$\rho(\omega) = -i(G_R(\omega) - G_A(\omega)) \tag{B.7}$$

We also note that the Euclidean correlation function in momentum space can be obtained from

$$G_E(\omega_l) = \begin{cases} G_R(i\omega_l) & l \geq 0 \\ G_A(i\omega_l) & l < 0 \end{cases}, \quad \omega_l = \frac{2\pi l}{\beta}, \quad l \in \mathbf{Z} \tag{B.8}$$

Some further remarks:

1. From (B.5)–(B.6),

$$\rho(-\omega) = -\rho(\omega), \quad G_{12}(-\omega) = G_{12}(\omega), \quad G_R(-\omega) = G_A(\omega). \quad (\text{B.9})$$

2. For a theory with a discrete spectrum, from (B.5), the spectral function $\rho(\omega)$ and $G_+(\omega)$ are given by a sum of discrete delta functions supported on the real axis, while $G_R(\omega)$ is given by a discrete sum of poles along the real axis.

B.2 Self-energy in real-time formalism

In this section we consider real-time correlation functions of fundamental fields M_α in perturbation theory using the real-time formalism. We denote various quantities in (B.1) with $\mathcal{O} = M_\alpha$ by $D_+^{(\alpha)}, D_F^{(\alpha)}$ etc and will suppress superscript α from now on. We prove that the corresponding spectral density functions have a discrete spectrum after the resummation of the self-energy diagrams à la Dyson. For simplicity, we will consider the high temperature limit so that we can ignore the singlet projection (see (3.4)).

In the real time formalism [30] the degrees of freedom of the theory get doubled (see also [51]). For each original field (type 1) in (2.1) one introduces an equivalent field (type 2)⁴⁶ whose interaction vertices differ by a sign from the ones for fields of type 1. Vertices therefore do not mix the two different kind of fields but propagators do and are written as a 2×2 matrix. For example, in frequency space the propagator for M_α (in the interacting theory) can be written as $D_{ab}(\omega)$, $a, b = 1, 2$ with each component given by

$$\begin{aligned} D_{11}(\omega) &= D_F(\omega), \\ D_{22}(\omega) &= D_{11}^*(-\omega) \\ D_{12}(\omega) &= \frac{e^{\frac{\beta}{2}\omega}}{e^{\beta\omega} - 1} \rho(\omega), \\ D_{21}(\omega) &= D_{12}(\omega) \end{aligned} \quad (\text{B.10})$$

D_{ab} can be diagonalized as

$$D_{ab} = U \begin{pmatrix} D_g(\omega) & 0 \\ 0 & D_g^*(\omega) \end{pmatrix} U \quad (\text{B.11})$$

with

$$\begin{aligned} U &= \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}, \\ \cosh\gamma &= \frac{e^{\frac{\beta}{2}|\omega|}}{\sqrt{e^{\beta|\omega|} - 1}}, \\ \sinh\gamma &= \frac{1}{\sqrt{e^{\beta|\omega|} - 1}} \end{aligned} \quad (\text{B.12})$$

⁴⁶In a path integral derivation these correspond to the fields whose time argument is $t - i\sigma$ and we will take $\sigma = \frac{\beta}{2}$.

and

$$D_g(\omega) = i \int \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega - \omega' + i\epsilon\omega} = \begin{cases} -iD_R(\omega) & \omega > 0 \\ r - iD_A(\omega) & \omega < 0 \end{cases} \quad (\text{B.13})$$

The last expression in (B.13) implies that when analytically continued from the positive real axis, $D_g(\omega)$ cannot have singularities in the upper half ω -plane. Similarly when analytically continued from the negative real axis, $D_g(\omega)$ cannot have singularities in the lower half ω -plane. Note that $D_g(\omega)$ can have a discontinuity at $Im(\omega) = 0$. If $D_g(\omega)$ does turn out to be analytic on the real axis, then it can have singularities only on the real axis in the limit $\epsilon \rightarrow 0$, which in turn implies that D_R, D_A and D_F can have singularities only on the real axis in the limit $\epsilon \rightarrow 0$.

We will now show that $D_g(\omega)$ obtained using the Dyson equation from any finite order computation of the self-energy is a rational function with singularities only on the real axis. This implies that the spectral function ρ consists of a sum of finite number of delta functions supported on the real axis.

Note that the Dyson equation can be written as

$$\frac{1}{D_g(\omega)} = \frac{1}{D_g^{(0)}(\omega)} - i\tilde{\Pi}(\omega) \quad (\text{B.14})$$

where

$$D_g^{(0)} = \frac{i}{\omega^2 - m^2 + i\epsilon} \quad (\text{B.15})$$

is the free theory expression and $\tilde{\Pi}(\omega)$ can be computed from perturbation theory as follows: (i) Compute 2×2 matrix $\Pi_{ab}(\omega)$ from the sum of amputated 1PI diagrams for the propagator in real time formalism; (ii) Diagonalizing $\Pi_{ab}(\omega)$ using (B.12), i.e.

$$\Pi_{ab} = U \begin{pmatrix} \tilde{\Pi}(\omega) & 0 \\ 0 & \tilde{\Pi}^*(\omega) \end{pmatrix} U. \quad (\text{B.16})$$

That Π_{ab} can be diagonalized using U is a consequence (B.11).

Now expanding D_g and $\tilde{\Pi}$ in power series of λ

$$\begin{aligned} D_g &= D_g^{(0)} + \lambda D_g^{(1)} + \lambda^2 D_g^{(2)} + \dots \\ \tilde{\Pi} &= \lambda \tilde{\Pi}^{(1)} + \lambda^2 \tilde{\Pi}^{(2)} + \dots \end{aligned} \quad (\text{B.17})$$

from equation (B.14) we have

$$D_g^{(1)} = D_g^{(0)}(i\tilde{\Pi}^{(1)})D_g^{(0)}, \quad D_g^{(2)} = D_g^{(0)}(i\tilde{\Pi}^{(2)})D_g^{(0)} + D_g^{(0)}(i\tilde{\Pi}^{(1)})D_g^{(0)}(i\tilde{\Pi}^{(1)})D_g^{(0)}, \quad \dots \quad (\text{B.18})$$

From our discussion in section 3 (applied to fundamental fields), at any finite order in perturbation theory $\rho(\omega)$ consists of sums of terms of the form (3.7). Plugging such a $\rho(\omega)$ into (B.13) one finds that $D_g^{(n)}(\omega)$ is a rational function and is analytic on the real axis at each order in the perturbative expansion (i.e. there is no discontinuity at $Im(\omega) = 0$). Using (B.18) we find that $\tilde{\Pi}^{(n)}(\omega)$ must also be a rational function and analytic on real axis. This in turn implies that the resummed $D_g(\omega)$ found from (B.14) is a rational function

and analytic on real axis. We conclude that the singularities of D_g must lie on the real axis and there are only a finite number of them at any finite order in the computation of the self-energy Π . From (B.13) the spectral density function must be a finite sum of delta functions supported on the real axis.

C. Energy spectrum and eigenvectors of sparse random matrices

In this appendix we summarize features of eigenvalues and eigenvectors of a random sparse matrix found in [52–54]. Consider an $M \times M$ real symmetric matrix A whose elements A_{ij} for $i \geq j$ are independent identically distributed random variables with even probability distribution $f(A_{ij})$. Let $f(x)$ be of the following form:

$$f(x) = (1 - \alpha)\delta(x) + \alpha h(x) \tag{C.1}$$

where $0 < \alpha < 1$ and $h(x)$ is even and not delta-function like at $x = 0$. Let the variance of $h(x)$ be v^2 . The parameter α measures the sparsity of the matrix: for each row or column of the matrix there will be on average $\alpha M = K$ elements which are different from zero. K is called the connectivity of the matrix. When $K < 1$ it is possible for eigenvectors to be localized in a subspace with dimension smaller than M . For $K > 1$ in the large M limit no such localization occurs and the matrix has to be diagonalized in the full M dimensional space.

When $K \gg 1$, the density of states reduces to Wigner’s semicircular law in an expansion in K^{-1} :

$$\rho(E) = \frac{1}{2\pi\Gamma^2} \sqrt{4\Gamma^2 - E^2} (1 + O(K^{-1})) \tag{C.2}$$

Where E is the eigenvalue value and Γ is given by:

$$\Gamma^2 = Kv^2 (1 + O(K^{-1})) \tag{C.3}$$

Notice that Kv^2 is the average value of

$$\Gamma_i^2 = \sum_{j \neq i} |A_{ij}|^2 \tag{C.4}$$

over the rows or columns of the sparse matrix. The first correction to $\rho(E)$ gives a change in the edge location, however there also are nonperturbative tails to the distribution which for $E \gg \Gamma$ assume the form:

$$\rho(E) \sim \left(\frac{E^2}{eK} \right)^{-E^2} \tag{C.5}$$

Their effect is to make the spectrum unbounded.

Denote by T the orthogonal change of basis matrix which brings A to diagonal form for $K \gg 1$. T has a random uniform distribution over the group of orthogonal $M \times M$ matrices. Therefore the eigenvectors of A are a random orthonormal basis of the total space which means that apart from correlations⁴⁷ which are negligible in the large M limit their elements are independently distributed gaussian random variables with mean 0 and variance $\frac{1}{M}$. In particular the eigenvectors are completely delocalized. Therefore for large K the situation is similar to that for the Gaussian Orthogonal Ensemble (GOE).

⁴⁷which are due to normalization conditions.

D. Single anharmonic oscillator⁴⁸

It is clear that the argument presented in section 4 applies to the real time correlation functions of a single anharmonic oscillator at finite temperature (with changes of combinatorial factors)

$$S = \int dt \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2 - \frac{1}{4!} \lambda x^4 \right) . \quad (\text{D.1})$$

For example one can conclude that the perturbation theory for

$$D_+(t) = \langle x(t)x(0) \rangle_\beta \quad (\text{D.2})$$

should diverge at a time scale (4.25) for $T \gg \omega_0$ (we set $\omega_0 = 1$ in (D.1)). Here we give an alternative derivation of this. Inserting complete sets of states in (D.2) we find that

$$D_+(t) = Z^{-1} \sum_{n,m} |\langle n|x|m \rangle|^2 e^{-\beta E_n - it(E_m - E_n)} \quad (\text{D.3})$$

where $Z = \sum_n e^{-\beta E_n}$ and $|n\rangle$ are interacting theory eigenstates. If we are interested only in contributions of the form $(\lambda t)^n$ we get:

$$D_+(t) = Z_0^{-1} \sum_{n,m} |\langle n|x|m \rangle_0|^2 e^{-\beta E_n^{(0)} - it(E_m^{(0)} - E_n^{(0)}) - it\lambda(E_m^{(1)} - E_n^{(1)})} + \dots \quad (\text{D.4})$$

where quantities with index 0 are computed in the free theory and $\lambda E^{(1)}$ are the energy shifts at first order in perturbation theory. Equation (D.4) can be evaluated as

$$D_+(t) = \frac{1}{2} Z_0^{-1} \sum_{n=0}^{\infty} (n+1) [e^{-\beta(n+\frac{1}{2}) - it(1+\frac{\lambda}{8}(n+1))} + e^{-\beta(n+\frac{3}{2}) + it(1+\frac{\lambda}{8}(n+1))}] + \dots \quad (\text{D.5})$$

which can be summed to give

$$D_+(t) = \frac{(e^\beta - 1)e^{\beta - it(1 - \frac{\lambda}{8})}}{2(e^{\beta + \frac{it\lambda}{8}} - 1)^2} + \frac{(e^\beta - 1)e^{it(1 - \frac{\lambda}{8})}}{2(e^{\beta - \frac{it\lambda}{8}} - 1)^2} + \dots \quad (\text{D.6})$$

In (D.6) there are double poles at

$$t = \pm i \frac{8\beta}{\lambda} + k \frac{16\pi}{\lambda}, \quad k \in Z . \quad (\text{D.7})$$

If one resums the diagrams discussed in section 4, one would then get simple poles and the positions of the poles are further away from the real axis than those of (D.6) indicating that there are some positive contributions not captured by the class of Feynman diagrams.

The reason for the behavior (D.5)–(D.7) can be attributed to the fact that the first order energy shift behaves as

$$\lambda(E_{n+1}^{(1)} - E_n^{(1)}) \propto \lambda n . \quad (\text{D.8})$$

⁴⁸This section is motivated from a discussion with Steve Shenker.

Thus when n is sufficiently large i.e. $n \sim \frac{1}{\lambda}$, perturbation theory breaks down due to level crossing. Also note that the divergence of perturbation theory at $t \sim \frac{1}{\lambda T}$ has nothing to do with the standard argument of the breakdown of perturbation theory by taking $\lambda \rightarrow -\lambda$. Indeed the behavior here is due to a single class of diagrams not to the $n!$ growth of the number of diagrams.

We emphasize that while from the Feynman diagram point of view the discussion for anharmonic oscillators is almost identical to that for a matrix quantum mechanics (except that for matrix quantum mechanics one restricts to planar diagrams), the underlying physics for the breakdown of perturbation theory appears to be very different:

1. In the example of a single anharmonic oscillator, perturbation theory is asymptotic, i.e. the n -th order expansion contains $n!$ independent diagrams. In contrast, in the planar expansion of a matrix quantum mechanics, the number of Feynmann diagrams grows only like a power in n . The class of planar diagrams we identified gives rise to $O(n!)$ contribution (in frequency space) at the n -th order. In the anharmonic oscillator example, given that the perturbative expansion is already divergent, one cannot really draw any clear conclusion from this class of diagrams. For instance, its contribution could be overwhelmed by those from $n!$ other diagrams. In contrast, in the case of matrix quantum mechanics, the contribution from the particular class of diagrams makes an otherwise convergent perturbative expansion divergent. Given that the nature of perturbation theory is very different between the single anharmonic oscillator and the planar matrix quantum mechanics, one should be very careful in drawing any conclusion when comparing them. In particular, the fact that the anharmonic oscillator has a discrete spectrum does *not* imply that in the matrix quantum mechanics case, the divergence of the subclass of diagrams and hence the breakdown of perturbation theory are *not* related to a possible underlying quasi-continuous spectrum.
2. As indicated earlier in this appendix for a single anharmonic oscillator, the divergent behavior of the class of Feynman diagrams considered in section 4 should have to do with level mixing for states of energy $O(1/\lambda)$. Applying the same technique to a matrix quantum mechanics, one again expects to relate the divergent behavior of the class of Feynman diagrams to the mixing of energy levels which dominate the thermal ensemble (i.e. with energy $O(N^2)$). More explicitly, let us write (D.4) for the matrix case as

$$D_+(t) = Z_0^{-1} \sum_n e^{-\beta E_n^{(0)}} \sum_m |\langle n|M|m \rangle_0|^2 e^{-it(E_m^{(0)} - E_n^{(0)}) - it\lambda(E_m^{(1)} - E_n^{(1)})} + \dots \quad (\text{D.9})$$

As we discussed in the main text, the sum over m in the above equation will involve an exponentially large number of states with free theory energies ranging over of order $O(N)$. A naive estimate of $E_m^{(1)} - E_n^{(1)}$ also gives order $O(N)$. Here unfortunately the story appears to be rather complicated and it appears it is not possible to extract a divergent time scale $1/\lambda T$ from (D.9).

In summary, the Feynman diagram argument demonstrates the breakdown of perturbation theory, but does not tell us why or how it breaks down. It is certainly possible that completely different mixing behaviors in the energy levels may be reflected similarly by Feynman diagrams. In the anharmonic oscillator example discrete levels mix, while in the matrix quantum mechanics a quasi-continuous spectrum mixes. One must be very careful in extrapolating the results for an anharmonic oscillator to a matrix quantum mechanics.

E. Estimate of various quantities

We now estimate (6.8)–(6.11) after averaging them over states of similar energies. We will be interested in how these quantities scale with N in the large N limit. An important property that we will assume below for these averaged quantities is that they are slow-varying functions of ϵ or E . In the large N limit, we can then estimate them using the corresponding thermal averages, which can in turn be expressed in terms of various correlation functions at finite temperature. For example, the thermal average of Σ_i is

$$\widehat{\Sigma}(\beta) = \frac{1}{Z} \sum_i e^{-\beta E_i} \Sigma_i = \frac{1}{Z} \int dE e^{-\beta E} \Omega(E) \Sigma(E) \quad (\text{E.1})$$

where $\Sigma(E)$ is the microcanonical average and $\Omega(E) = e^{S(E)}$ the density of states. Since $\Sigma(E)$ is a slow-varying function of E , we can perform a saddle point approximation of the last expression, yielding

$$\Sigma(E) \approx \widehat{\Sigma}(\beta_E) (1 + O(1/N^2)) \quad (\text{E.2})$$

with β_E determined by $\frac{\partial S(E)}{\partial E} = \beta_E$. Using the last equality of (6.11) we can write $\widehat{\Sigma}(\beta)$ as

$$\begin{aligned} \widehat{\Sigma}(\beta) &= \frac{1}{Z} \sum_{i,j,i \neq j} e^{-\beta E_i} |\langle i|V|j \rangle|^2 \\ &= \langle V(0)V(0) \rangle_\beta \end{aligned} \quad (\text{E.3})$$

where $\langle V(0)V(0) \rangle_\beta$ denotes the connected Wightman function as defined by (B.3).

From the standard large N scaling argument (E.3) is of order $O(N^2)$ (recall that we include a factor of N in the definition of V). Thus unless (E.3) is zero at leading order we conclude that $\Sigma(E)$ can be written in a form

$$\Sigma(E) = N^2 h(\lambda, E/N^2) \quad (\text{E.4})$$

where $h(\lambda, \mu)$ is a function independent of N . An exactly parallel argument can be applied to $\sigma(\epsilon)$ in which case (E.3) is replaced by expectation values in free theory and thus we find that

$$\sigma(\epsilon) = N^2 \widetilde{h}(\lambda, \epsilon/N^2) \quad (\text{E.5})$$

for some function \widetilde{h} .

As another example, let us look at the thermal average of (6.10),

$$\frac{1}{Z} \sum_i e^{-\beta E_i} \bar{\epsilon}_i = \frac{1}{Z} \int dE e^{-\beta E} \Omega(E) \bar{\epsilon}(E) \approx \bar{\epsilon}(E_\beta) \quad (\text{E.6})$$

Using the last equality of (6.10), the left hand side of (E.6) can in turn be written as

$$E_\beta - \langle V \rangle_\beta \tag{E.7}$$

where $\langle V \rangle_\beta$ is the thermal one-point function of V in the interacting theory and scales with N as $O(N^2)$. Thus we can write

$$\bar{\epsilon}(E) = N^2 g(\lambda, E/N^2) \tag{E.8}$$

for some function h . An exactly parallel argument yields

$$\bar{E}(\epsilon) = N^2 \tilde{g}(\lambda, \epsilon/N^2) . \tag{E.9}$$

To summarize, we find that the averaged values of $\Gamma(\epsilon)$ and $\Delta(E)$ are both of order $O(N)$ in the 't Hooft limit for any nonzero λ . Thus in the large N limit, both the correlation length between interacting theory energy levels and the energy range that the free theory states are mixed under perturbation go to infinity.

F. Some useful relations

In this appendix we derive some important relations which will be used in appendix G to derive the matrix elements of an operator \mathcal{O} between generic states in the high energy sector.

F.1 Density of states

The conservation of states implies that the density of states $\Omega(E)$ of the full theory and $\Omega_0(\epsilon)$ of the free theory should be related by

$$\Omega(E) = \Omega_0(\bar{\epsilon}(E)) \frac{d\bar{\epsilon}(E)}{dE} \tag{F.1}$$

which implies

$$\frac{1}{\Omega(E)} \frac{d\Omega(E)}{dE} = \frac{d\bar{\epsilon}(E)}{dE} \frac{1}{\Omega_0(\bar{\epsilon}(E))} \frac{d\Omega_0}{d\epsilon} \Big|_{\bar{\epsilon}(E)} + \frac{\frac{d^2\bar{\epsilon}(E)}{dE^2}}{\frac{d\bar{\epsilon}(E)}{dE}}$$

In the large N limit the second term in the above equation should be of order $O(1/N^2)$. Thus we find that

$$\beta(E) = \beta_0(\bar{\epsilon}(E)) \frac{d\bar{\epsilon}(E)}{dE} \tag{F.2}$$

with

$$\beta(E) = \frac{1}{\Omega(E)} \frac{d\Omega(E)}{dE}, \quad \beta_0(\epsilon) = \frac{1}{\Omega_0(\epsilon)} \frac{d\Omega_0}{d\epsilon} . \tag{F.3}$$

We also expect that

$$\bar{\epsilon}(\bar{E}(\epsilon)) \approx \epsilon \tag{F.4}$$

Note that all the above relations are valid only to leading order in N .

F.2 Properties of $\chi_E(\epsilon)$ and $\rho_\epsilon(E)$

Consider the microcanonical average of (6.5) and (6.6), which we denote as $\rho_\epsilon(E)$ and $\chi_E(\epsilon)$ respectively. From (6.5) and (6.6) one should have

$$\rho_\epsilon(E) = \frac{\Omega(E)}{\Omega_0(\epsilon)} \chi_E(\epsilon) . \quad (\text{F.5})$$

From (6.7) we should also have

$$\int d\epsilon \chi_E(\epsilon) = 1 \quad (\text{F.6})$$

and

$$\int dE \rho_\epsilon(E) = \frac{1}{\Omega_0(\epsilon)} \int dE \Omega(E) \chi_E(\epsilon) = 1 . \quad (\text{F.7})$$

Given that

$$\bar{\epsilon}(E) = \int d\epsilon \epsilon \chi_E(\epsilon), \quad \Sigma(E) = \Delta^2(E) = \int d\epsilon (\epsilon - \bar{\epsilon}(E))^2 \chi_E(\epsilon) \quad (\text{F.8})$$

we can write $\chi_E(\epsilon)$ as

$$\chi_E(\epsilon) = f_E(\epsilon - \bar{\epsilon}(E)) \quad (\text{F.9})$$

with f_E a function which has a spread of $\Delta(E) \sim O(N)$. Since we expect $f_E(\omega)$ to fall off quickly to zero in the large N limit outside the range $(-\frac{1}{2}\Delta(E), \frac{1}{2}\Delta(E))$, equations (F.6) and (F.7) lead to

$$\int_{-\infty}^{\infty} d\omega f_E(\omega) = 1 \quad (\text{F.10})$$

and

$$\int_{-\infty}^{\infty} dE \frac{\Omega(E)}{\Omega_0(\epsilon)} f_E(\epsilon - \bar{\epsilon}(E)) = 1 \quad (\text{F.11})$$

Changing the integration variable of (F.11) to $\epsilon' = \bar{\epsilon}(E)$ and using (F.4), we find that

$$\int d\epsilon' \frac{\Omega_0(\epsilon')}{\Omega_0(\epsilon)} f_{\bar{E}(\epsilon')}(\epsilon - \epsilon') = \int_{-\infty}^{\infty} d\omega f_{\bar{E}(\epsilon)}(\omega) \frac{\Omega_0(\epsilon - \omega)}{\Omega_0(\epsilon)} = 1 \quad (\text{F.12})$$

where in the second expression we have replaced $f_{\bar{E}(\epsilon')}$ by $f_{\bar{E}(\epsilon)}$. This is because, as a function of $\epsilon - \epsilon'$, the spread of f is of order $O(N)$, while $\bar{E}(\epsilon') \approx \bar{E}(\epsilon) + O(\frac{\epsilon' - \epsilon}{N^2}) \approx \bar{E}(\epsilon)$. The second expression of (F.12) can now be written as

$$\int_{-\infty}^{\infty} d\omega f_E(\omega) e^{-\beta_0(\bar{\epsilon}(E))\omega} = 1 \quad (\text{F.13})$$

Equations (F.10) and (F.13) can be written in a more symmetric manner as

$$\int_{-\infty}^{\infty} d\omega e^{\frac{1}{2}\beta(E)\omega} g_E(\omega) = \int_{-\infty}^{\infty} d\omega e^{-\frac{1}{2}\beta(E)\omega} g_E(\omega) = 1 \quad (\text{F.14})$$

where we have introduced a function

$$g_E(\omega) = e^{-\frac{1}{2}\beta(E)\omega} \frac{d\bar{\epsilon}(E)}{dE} f_E \left(\frac{d\bar{\epsilon}(E)}{dE} \omega \right) . \quad (\text{F.15})$$

Equations (F.14) imply that $g_E(\omega)$ should fall off faster than $e^{-\frac{1}{2}\beta(E)|\omega|}$ as $\omega \rightarrow \pm\infty$.

F.3 A relation between matrix elements and correlation functions in free theory

In this subsection we derive in free theory a relation between the matrix elements of an operator \mathcal{O} between states in the high energy sector and correlation functions. For simplicity we consider theories with a single fundamental frequency ω_0 , like $\mathcal{N} = 4$ SYM or (2.3).

The Lehmann spectral decomposition for frequency space Wightman function $G_+^{(0)}(\omega)$ of some operator \mathcal{O} in free theory can be written as

$$G_+^{(0)}(\omega) = \frac{1}{Z_0} \sum_{a,b} e^{-\beta\epsilon_a} \rho_{ab} \delta(\omega - \epsilon_b + \epsilon_a) \quad (\text{F.16})$$

where $\rho_{ab} = |\langle a|\mathcal{O}(0)|b\rangle|^2$. Due to energy conservation, \mathcal{O} can only connect levels whose energy differences lie between $-\Delta\omega_0$ and $\Delta\omega_0$, where Δ is the dimension of \mathcal{O} , i.e. ρ_{ab} can only be non-vanishing for $|\epsilon_a - \epsilon_b| \leq \Delta\omega_0$. We can thus rewrite (F.16) as

$$G_+^{(0)}(\omega) = \frac{1}{Z_0} \sum_{k=-\Delta}^{\Delta} G_k \delta(\omega - k\omega_0) \quad (\text{F.17})$$

with

$$\begin{aligned} G_k &= \frac{1}{Z_0} \sum_a e^{-\beta\epsilon_a} \sum_{\epsilon_b = \epsilon_a + k\omega_0} \rho_{ab} \\ &= \frac{1}{Z_0} \sum_a e^{-\beta\epsilon_a} \rho_k(a) \end{aligned} \quad (\text{F.18})$$

where $\sum_{\epsilon_b = \epsilon}$ denotes that one sums over $|b\rangle$ whose energy is given by $\epsilon_b = \epsilon$. Note here we have assumed that the free theory energy levels are equally spaced as in $\mathcal{N} = 4$ SYM theory on S^3 . We also introduced

$$\rho_k(a) = \sum_{\epsilon_b = \epsilon_a + k\omega_0} \rho_{ab} \quad (\text{F.19})$$

We now separate the sum a in (F.18) in terms of energies and degeneracies, i.e.

$$\sum_a = \sum_{\epsilon} \sum_{\epsilon_a = \epsilon}$$

We thus find that

$$G_k = \frac{1}{Z_0} \sum_{\epsilon} \mathcal{N}(\epsilon) e^{-\beta\epsilon} \bar{\rho}_k(\epsilon) \quad (\text{F.20})$$

where we have introduced the micro-canonical average of $\rho_k(b)$ for energy ϵ

$$\bar{\rho}_k(\epsilon) = \frac{1}{\mathcal{N}(\epsilon)} \sum_{a \in \epsilon} \rho_k(a) = \frac{1}{\mathcal{N}(\epsilon)} \sum_{\epsilon_a = \epsilon} \sum_{\epsilon_b = \epsilon + k\omega_0} \rho_{ab} \quad (\text{F.21})$$

We expect that the microcanonical average $\bar{\rho}_k(\epsilon)$ should be a slow varying function of ϵ , i.e. it can be written in a form $N^\alpha f(\epsilon/N^2)$ for some constant α . In the large N

limit since $\mathcal{N}(\epsilon)e^{-\beta\epsilon}$ is sharply peaked at ϵ_β specified by, one can perform a saddle point approximation in (F.20) to get

$$G_k = \bar{\rho}_k(\epsilon_\beta) + \dots \quad (\text{F.22})$$

From (F.16) we thus find that

$$G_+(\omega) = \sum_k \bar{\rho}_k(\epsilon_\beta) \delta(\omega - k\omega_0) \quad (\text{F.23})$$

In the large N limit since the connected part of $G_+(\omega)$ scales with N as $O(N^0)$, thus we find from (F.23) that

$$\bar{\rho}_k(\epsilon_\beta) \sim O(N^0). \quad (\text{F.24})$$

G. Derivation of matrix elements

In this appendix we give a derivation of (6.14). The main object of interests to us is

$$\begin{aligned} \rho_{ij} &= \sum_{a,b} |c_{ia}|^2 |c_{jb}|^2 \rho_{ab} \\ &= \sum_a |c_{ia}|^2 \sum_k \sum_{\epsilon_b = \epsilon_a + k\omega_0} |c_{jb}|^2 \rho_{ab} \end{aligned} \quad (\text{G.1})$$

Due to the sparse and random nature of ρ_{ab} , one cannot naively approximate the sums over a and b in by integrals. Instead one must be careful with the discreteness nature of the sum. Note that

$$\sum_{\epsilon_b = \epsilon_a + k\omega_0} |c_{jb}|^2 \rho_{ab} \approx \bar{c}_j(\epsilon_a + k) \sum_{\epsilon_b = \epsilon_a + k\omega_0} \rho_{ab} = \bar{c}_j(\epsilon_a + k) \rho_k(a)$$

which can be justified as follows. Inside a given energy shell, ρ_{ab} can be treated as a random sparse matrix. Thus one can treat the summand as a random sampling of $|c_{jb}|^2$. Since the number of sampling points goes to infinity (as a power in N) in the large N limit, we can approximate $|c_{jb}|^2$ by its average value of the energy shell. We now have

$$\begin{aligned} \rho_{ij} &= \sum_k \sum_a |c_{ia}|^2 \bar{c}_j(\epsilon_a + k) \rho_k(a) \\ &= \sum_k \sum_\epsilon \bar{c}_j(\epsilon + k) \sum_{\epsilon_a = \epsilon} |c_{ia}|^2 \rho_k(a) \\ &= \sum_k \sum_\epsilon \bar{c}_j(\epsilon + k) \mathcal{N}(\epsilon) \bar{c}_i(\epsilon) \bar{\rho}_k(\epsilon) \end{aligned} \quad (\text{G.2})$$

In the second line above we separated the sum over all states a into the sum over the energy and the sum over states in each energy shell. In the third line we replaced the sum in an energy shell by its average values. The replacement is all right since $|c_{ia}|^2$ and $\rho_k(a)$ are completely independent variables, so the average of their product should factorize.

Now given that all quantities in the last line of (G.2) are averaged quantities, we approximate the sum over ϵ by an integral. Averaging i, j over states of the same energy and using (6.6), we find

$$\begin{aligned}
 \rho_{E_1 E_2} &= \sum_k \int \frac{d\epsilon}{\Omega_0(\epsilon+k)} \chi_{E_2}(\epsilon+k) \chi_{E_1}(\epsilon) \bar{\rho}_k(\epsilon) \\
 &= \sum_k \int \frac{d\epsilon}{\Omega_0(\epsilon+k)} f_{E_2}(\epsilon+k-\bar{\epsilon}_2) f_{E_1}(\epsilon-\bar{\epsilon}_1) \bar{\rho}_k(\epsilon) \\
 &= \sum_k \int \frac{dp}{\Omega_0(\bar{\epsilon}_{12}+p+k)} f_{E_2}(p+k+\frac{1}{2}\Delta_{12}) f_{E_1}(p-\frac{1}{2}\Delta_{12}) \bar{\rho}_k(p+\bar{\epsilon}_{12}) \quad (\text{G.3})
 \end{aligned}$$

with

$$\bar{\epsilon}_{1,2} = \bar{\epsilon}(E_{1,2}), \quad \bar{\epsilon}_{12} = \frac{1}{2}(\bar{\epsilon}(E_1) + \bar{\epsilon}(E_2)) = \bar{\epsilon}(E), \quad \Delta_{12} = \bar{\epsilon}(E_1) - \bar{\epsilon}(E_2) = \left. \frac{d\bar{\epsilon}(E)}{dE} \right|_E \omega$$

where

$$E = \frac{E_1 + E_2}{2}, \quad \omega = E_1 - E_2$$

Equation (G.3) can be further simplified as

$$\begin{aligned}
 \rho_{E_1 E_2} &= \frac{1}{\Omega(E)} \sum_k G_{12}(k) \int_{-\infty}^{\infty} dp g_E\left(p+k'+\frac{1}{2}\omega\right) g_E\left(p-\frac{1}{2}\omega\right) \\
 &= \frac{1}{\Omega(E)} A(\omega; E) \quad (\text{G.4})
 \end{aligned}$$

with

$$G_{12}(k) = e^{-\frac{1}{2}\beta_0(\bar{\epsilon}_{12})k} \bar{\rho}_k(\bar{\epsilon}_{12}), \quad k' = \frac{1}{\frac{d\bar{\epsilon}(E)}{dE}} k \quad (\text{G.5})$$

and $g_E(\omega)$ was defined in (F.15). Note that from equation (F.23), G_{12} are essentially the Fourier components of free theory correlation functions. $A(\omega; E)$ should be a smooth function of ω since the integral in (G.4) appears to be well defined for all ω . It is easy to check that

$$A(-\omega; E) = A(\omega; E) \quad (\text{G.6})$$

since $G_{12}(k) = G_{12}(-k)$. Further as $\omega \rightarrow \infty$, we find that

$$A(\omega; E) \propto e^{-\frac{1}{2}\beta(E)|\omega|} . \quad (\text{G.7})$$

Now let us examine possible singularities of $A(\omega, E)$ on the real axis. We start with the definition (F.9) of f_E . Since $\chi_E(\epsilon)$ is the average of (6.6) over states of similar energies, $f_E(\omega)$ must be a real positive function of $\omega \in R$. Then the function $g_E(\omega)$ introduced in (F.15) should also be real and positive as $\bar{\epsilon}(E)$ is a monotonous function of E . The positivity and normalization conditions (F.14) imply that $g_E(\omega)$ can at most have integrable singularities of the form⁴⁹

$$g_E(\omega) \approx \frac{K_i}{|\omega - \omega_i|^{\alpha_i}}, \quad \omega \rightarrow \omega_i, \quad \alpha_i < 1 \quad (\text{G.8})$$

⁴⁹Such integrable singularities can only arise if c_{ia} have accumulation points in the large N limit. While it appears unlikely that this can happen, we do not have a rigorous proof at the moment.

Note that the closer α_i is to one the smaller is K_i from the normalization requirement. Now let us look at the definition (G.4) of $A(\omega; E)$,

$$A(\omega; E) = \sum_k G_{12}(k) s(\omega + k) \tag{G.9}$$

where

$$s(\omega) = \int_{-\infty}^{\infty} dx g_E\left(x + \frac{1}{2}\omega\right) g_E\left(x - \frac{1}{2}\omega\right) . \tag{G.10}$$

Note that the finite sum over k in (G.9) cannot introduce singularities in ω therefore we focus on $s(\omega)$. As $g_E(\omega)$ falls off faster than $e^{-\frac{1}{2}\beta(E)|\omega|}$ as $\omega \rightarrow \pm\infty$, the integral in (G.10) is convergent for $x \rightarrow \pm\infty$. Thus we only need to worry about possible divergences arising from the middle of the integration range. Integrating (G.10) we find that

$$\int_{-\infty}^{\infty} d\omega s(\omega) = \int_{-\infty}^{\infty} dx g_E(x) \int_{-\infty}^{\infty} dy g_E(y) \tag{G.11}$$

which is finite by (F.14) . Therefore the only singularities allowed for $s(\omega)$ are of integrable kind $\frac{K}{|\omega - \omega_s|^\alpha}$ with $\alpha < 1$. We can find the locations of ω_s in terms of (integrable) singularities of $g_E(\omega)$ as follows. Since $g_E(x - \frac{1}{2}\omega)$ and $g_E(x + \frac{1}{2}\omega)$ are both integrable the only possible divergences of (G.10) are at values of ω for which the integrable singularities of two function sit on top of each other. This happens for $\omega = \omega_i - \omega_j$ where the ω_i are the locations of the singularities for $g_E(\omega)$. For $\omega = \omega_i - \omega_j + \epsilon$ with ϵ small the integral (G.10) near $x \approx \frac{1}{2}(\omega_i + \omega_j)$ can be written as $K_i K_j \int_{-\delta}^{\delta} dy \frac{1}{|y - \frac{1}{2}\epsilon|^{\alpha_j} |y + \frac{1}{2}\epsilon|^{\alpha_i}}$ where δ is some multiple of ϵ . By rescaling we see that it behaves as $\epsilon^{1 - \alpha_i - \alpha_j}$. Therefore the integral $s(\omega)$ can at most have a singularity of the form $\frac{K_i K_j}{|\omega - \omega_i + \omega_j|^\alpha}$ with $\alpha = \alpha_i + \alpha_j - 1 < 1$.

Thus we conclude that on the real axis $A(\omega; E)$ can have at most integrable singularities of the form

$$A(\omega; E) \propto \frac{1}{|\omega - \omega_s|^{\alpha_s}}, \quad \alpha_s < 1 . \tag{G.12}$$

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